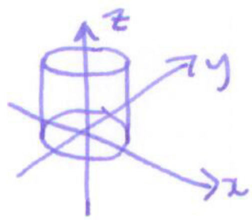
Describing regions

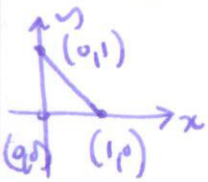
region $R \leftrightarrow$ limits in terms of x, y, z in same order

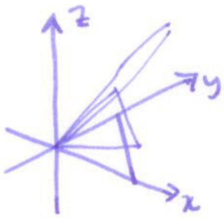
$R =$ cylinder
$0 \le z \le 1$
$x^2 + y^2 \le 1$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1} f(x,y,z)\, dz\, dy\, dx$$

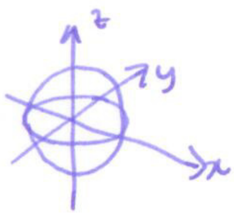
$R =$ region between planes $z = x+y$ and $z = 3x+5y$ over the triangle





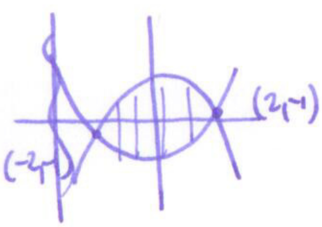
$$\int_{0}^{1} \int_{0}^{1-x} \int_{x+y}^{3x+5y} f(x,y,z)\, dz\, dy\, dx$$

$R =$ sphere $x^2 + y^2 + z^2 \le 1$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

<u>Example</u> ① find the area of the region between $y = 3 - x^2$ and $y = x^2 - 5$

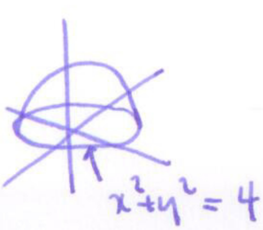


find intersections: $3 - x^2 = x^2 - 5 \quad 2x^2 = 8 \quad x = \pm 2$

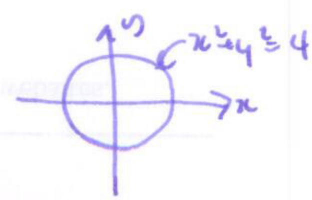
$$\int_{-2}^{2} \int_{x^2-5}^{3-x^2} 1\, dy\, dx = \Big[y \Big]_{x^2-5}^{3-x^2} = 3 - x^2 - (x^2 - 5)$$
$$= 8 - 2x^2.$$

$$\int_{-2}^{2} 8 - 2x^2\, dx = \left[8x - \tfrac{2}{3}x^3 \right]_{-2}^{2} = 16 - \frac{16}{3} - \left(-16 + \frac{16}{3} \right) = \frac{64}{3}.$$

② find the volume of the region below $z = 4 - x^2 - y^2$ and above the xy-plane



$$\int_{-2}^{+2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} 1\, dz\, dy\, dx$$



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} 1 \, dz\,dy\,dx \qquad \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} 4-x^2-y^2 \, dy\,dx \qquad \int_{-2}^{2} \frac{4}{3}(4-x^2)^{3/2} dx$$

try: $x = 2\sin\theta \quad \frac{dx}{d\theta} = 2\cos\theta \ldots$

$$\int_{-\pi/2}^{\pi/2} \frac{4}{3}\left(4-4\sin^2\theta\right)^{3/2} 2\cos\theta \, d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2} \cos^3\theta \cos\theta \, d\theta$$

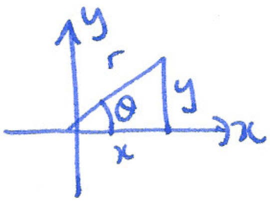
$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta \qquad \text{parts, or} \quad \cos^2\theta = \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}.$$

$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right) d\theta = \frac{16}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos 4\theta + \frac{1}{2} + 2\cos 2\theta + 1 \, d\theta$$

$$= \frac{16}{3}\left[-\frac{1}{8}\sin 4\theta + \frac{3\theta}{2} - \sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{16}{3}\cdot\frac{3}{2}\cdot\frac{2\pi}{2} = 8\pi.$$

§12.7 Cylindrical and spherical coordinates

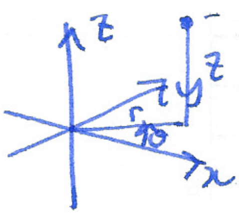
polar coords:



$r^2 = x^2+y^2 \qquad x = r\cos\theta$

$\tan\theta = y/x \qquad y = r\sin\theta$

cylindrical coords:

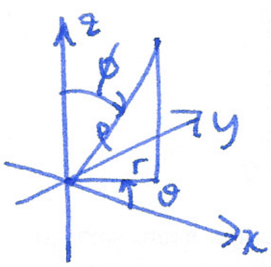


$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$

$z = z \qquad z = z$

spherical coords:

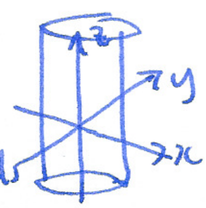


$x = \rho\cos\theta\sin\phi \qquad \rho^2 = x^2+y^2+z^2$

$y = \rho\sin\theta\sin\phi \qquad \theta = \tan^{-1}(y/x)$

$z = \rho\cos\phi \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

some sets are easier to describe in these coordinates:



$x^2+y^2 \le 1$

$r^2 \le 1$ in cylindricals

$x^2+y^2+z^2 \le 1$

$\rho^2 \le 1$ in sphericals.