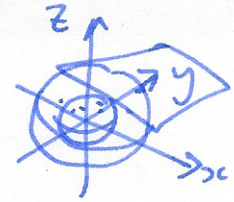
solve $\nabla f = \lambda \nabla g$ $\quad \nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 1, 2, 3 \rangle$




$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \\ 2z &= 3\lambda \end{aligned}\right\} \quad \left.\begin{aligned} x &= \lambda/2 \\ y &= \lambda \\ z &= \tfrac{3}{2}\lambda \end{aligned}\right\} \quad \frac{\lambda}{2} + 2\lambda + 3\cdot\frac{3}{2}\lambda = 4 \qquad \lambda = \frac{4}{7}$$

$x + 2y + 3z = 4$

so point is $\left\langle \frac{2}{7}, \frac{4}{7}, \frac{6}{7} \right\rangle$

<u>Example</u> minimize $f(x,y,z) = x^2y^2 + z^2$ subject to $x + y = 2$

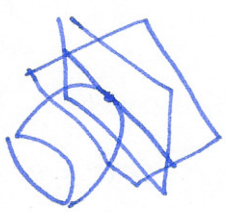
<u>and</u> $y + z = 4$

solve: $\nabla f = \lambda \nabla g + \mu \nabla h$

<u>why</u>?


extreme points when level sets of $f$ tangent to $g=0 \cap h=0$

i.e. tangent direction to $g=0 \cap h=0 \subset$ tangent plane to $f=c$



i.e. normal vector to $f=c$ is a sum of normal vectors to $g=0$ and $h=0$

$\nabla f = \langle 2xy^2, 2x^2y, 2z \rangle$

$\nabla g = \langle 1, 1, 0 \rangle$

$\nabla h = \langle 0, 1, 1 \rangle$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$$\left.\begin{aligned} 2xy^2 &= \lambda \\ 2x^2y &= \lambda + \mu \\ 2z &= \mu \end{aligned}\right\} \quad 2x^2y = 2xy^2 - 2z$$

$$\left.\begin{aligned} x + y &= 2 \\ y + z &= 4 \end{aligned}\right\} \quad \begin{aligned} y &= 2 - x \\ z &= 4 - y \end{aligned}$$

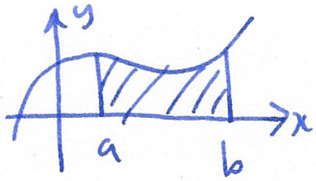
$2x^2(2-x) = 2x(2-x)^2 - 2(2+x)$ cubic find approx sol'ns $x = 0.3$

$y = 2.3$

$z = 1.7$.

# § 15.1 Integration in many variables

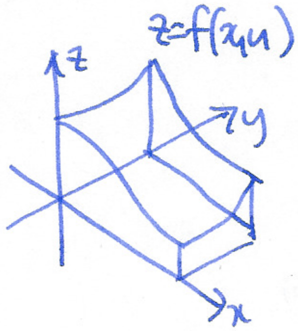
**1 var**



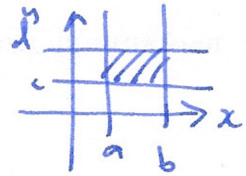

$\int_a^b f(x)\,dx$ = area under curve between $x=a$ and $x=b$

↑ compute this by finding the anti-derivative $F'(x)=f(x)$, then $\int_a^b f(x)\,dx = F(b)-F(a)$

**2 vars**



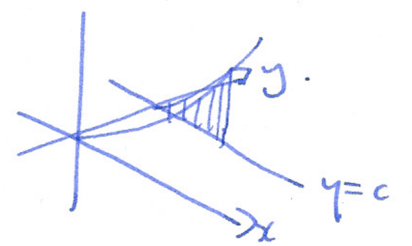

$\int_a^b \int_c^d f(x,y)\,dy\,dx$ = volume under surface over the region $c \leq y \leq d$, $a \leq x \leq b$




**recall** $\frac{\partial f}{\partial x}$ = "diff wrt $x$ keeping $y$ constant"

$\int_a^b f(x,y)\,dx$ = "integrate wrt $x$ keeping $y$ constant"

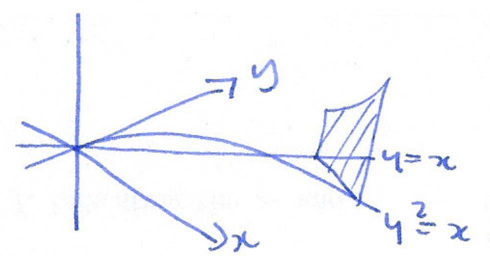
**Example** $\int_0^1 xy\,dx = \left[\frac{1}{2}x^2y\right]_0^1 = \frac{1}{2}y$




**Note** ① there should be no $x$'s remaining!

② the limits can depend on $y$!

**Example** $\int_y^{y^2} xy\,dx = \left[\frac{1}{2}x^2y\right]_y^{y^2} = \frac{1}{2}y^5 - \frac{1}{2}y^3$



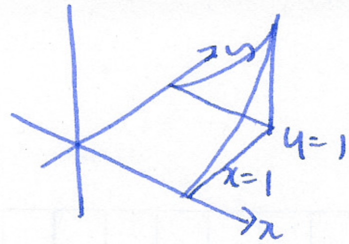

Q: what does this mean? $\int_0^1 xy\,dx$ means area between $x=0$, $x=1$

$\int_y^{y^2} xy\,dx$ means area between $x=y$ and $x=y^2$.

**Double integrals**

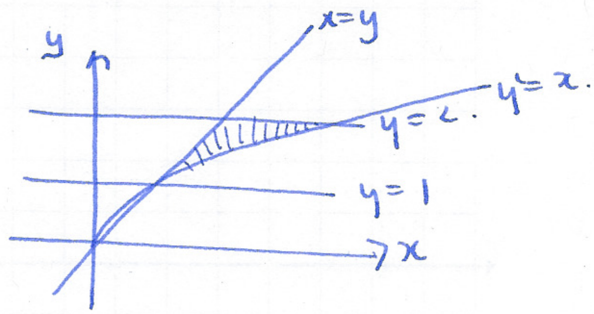
**Ex** $\int_0^1 \int_0^1 xy\,dx\,dy = \int_0^1 \frac{1}{2}y\,dy = \left[\frac{1}{4}y^2\right]_0^1 = \frac{1}{4}$ = volume under surface above region.

describes region $0 \le x \le 1$
$0 \le y \le 1$




② $\underbrace{\int_1^2 \underbrace{\int_y^{y^2}}\, xy \underbrace{dx}\, \underbrace{dy}}$ $= \int_1^2 \frac{1}{2}y^5 - \frac{1}{2}y^3\, dy = \left[\frac{1}{12}y^6 - \frac{1}{8}y^4\right]_1^2 =$ same number.

↑ describes region $1 \le y \le 2$
$y \le x \le y^2$



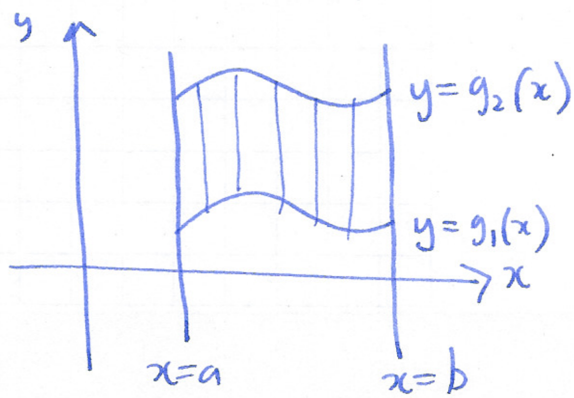

notation $\iint_D f(x,y)\, dx\, dy$

↑ name of region

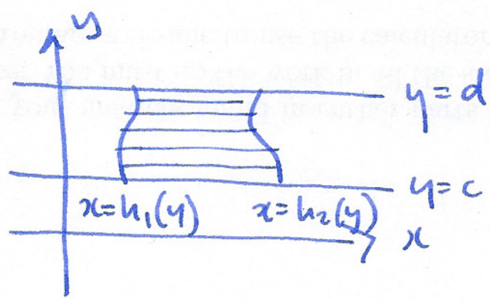
or $\iint_D f(x,y)\, dA$

nice regions $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$

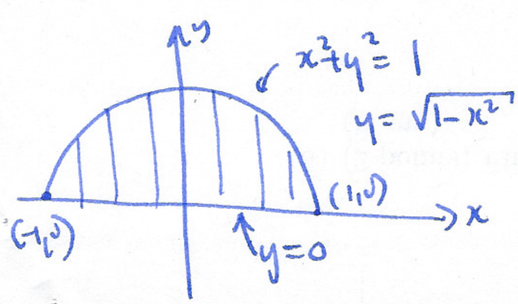
$a$, $b$ must be constants. $g_1(x)$, $g_2(x)$ may be functions of $x$




$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$




Example $D=$ upper half disc




$$\int_{-1}^{+1} \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

$+1$ ← biggest value of $x$

$\sqrt{1-x^2}$ ← top boundary

$0$ ← bottom boundary

$-1$ ← smallest value of $x$