$f(x,y) = (x^2+y^2)e^{-2x}$ $\quad f_x = 2xe^{-2x} + (x^2+y^2)e^{-2x}\cdot(-2)$

$$= e^{-2x}\left(2x - 2x^2 - 2y^2\right)$$

$$f_y = 2ye^{-2x}$$

find critical points: $\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \Rightarrow y = 0 \Rightarrow 2x - 2x^2 = 0 \quad 2(x)(x-1) = 0$

$x = 0, 1$

$(0,0), (1,0)$ critical points

$f_{xx} = -2e^{-2x}(2x-2x^2-2y^2) + e^{-2x}(2-4x) + \cancel{4e^{-2x}(4y)}$

$f_{xy} = \cancel{4e^{2x}} - 4ye^{-2x}$

$f_{yy} = 2e^{-2x}$

$D(0,0) = 2\cdot 2 - (-4\cdot 0) = 4 > 0$ min.

$D(1,0) =$

$(-2e^{-2}(0-0) + e^{-2}(-2))(2e^{-2}) - 0$

$= -4e^{-4} < 0$ saddle.

Absolute max/min

1 var: A cts function on a closed interval has an absolute max and min

$\leftarrow$ absolute max/min occurs at a max/min or endpoint.

2 vars $\quad D \subseteq \mathbb{R}^2 \quad f: \mathbb{R}^2 \to \mathbb{R}$

D is bounded if $D \subset$ disc of radius R about $(0,0)$.

$x \in D$ is an interior point if there is a small disc around x contained in D

otherwise x is a boundary point.

interior of D = union of all interior points

boundary of D = union of all boundary points

D is closed if D contains all of its boundary points

D is open if every point of D is an interior point

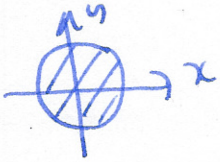
Examples

$[a,b]$ closed $\qquad (a,b)$ open $\qquad \mathbb{R}$ closed and open $\qquad [a, b)$ neither open nor closed

Thm $f(x,y)$ cts on $D \subseteq \mathbb{R}^2$ closed, bounded, then $f(x,y)$ has abs max/min on D.

Extreme ~~points~~ values occur either at critical points in the interior, or on the boundary

Example find the abs max/min of $f(x,y) = xy$ on the unit disc $x^2+y^2 \leq 1$



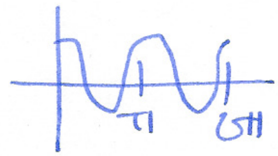
① find critical points

$$\left.\begin{aligned} f_x &= y \\ f_y &= x \end{aligned}\right\} \quad (0,0) \text{ only critical point.}$$

② find max/min on boundary:

parameterize boundary $(\cos\theta, \sin\theta)$ $0 \leq \theta \leq 2\pi$

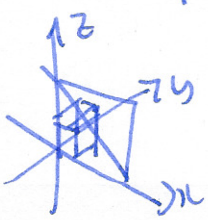
$f(\theta) = f(x(\theta), y(\theta)) = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$



$f'(\theta) = \cos 2\theta$ solve $f'(\theta) = 0$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

correspond to points $\to \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$.

Example find the box of max volume bounded by the coordinate planes, and the plane $x+y+z=1$

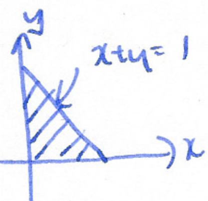


sides of box x, y, z, with $x+y+z=1$.

volume of box $V = xyz$

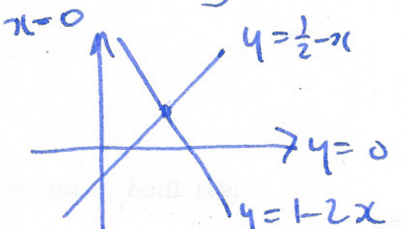
so sides $x, y, z = 1-x-y$

$V = xy(1-x-y) \leftarrow$ find max of this on $x \geq 0$, $y \geq 0$, $z \geq 0$ $1-x-y \geq 0$, $x+y \leq 1$.



① find critical points $V_x = y - 2xy - y^2$, $V_y = x - x^2 - 2xy$

solve $\left.\begin{aligned} V_x &= 0 \\ V_y &= 0 \end{aligned}\right\}$ ①: $y(1-2x-y) = 0$ ②: $x(1-x-2y) = 0$



($x=0$, $y=0$ on boundary)

lines intersect at $\left.\begin{aligned} 2x+y &= 1 \\ x+2y &= 1 \end{aligned}\right\}$ ①−2②: $-3y-1$ $y = 1/3$ $x = 1/3$.

$(1/3, 1/3)$ critical point.

② check max on boundary

$x=0 \quad 0 \le y \le 1 \quad V=0$

$y=0 \quad 0 \le x \le 1 \quad V=0$

$(t, t-1) \quad 0 \le t \le 1 \quad V = t(1-t)(1-t-(1-t)) = 0$

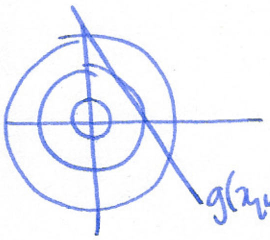
} $V=0$ on boundary

so max is $V(1/3, 1/3) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}$.

§14.8 Lagrange multipliers

optimization with constraints, i.e. max $f(x,y)$ subject to $g(x,y)=0$

Example ~~maximize~~ minimize $f(x,y) = \sqrt{x^2+y^2}$ subject to $\underbrace{x+y-4}_{g(x,y)}=0$



consider level sets $f(x,y)=c$

key point: at the extreme values of $f(x,y)$ on $g(x,y)=0$, the level sets of f are parallel to g

so we want to look for points where

level sets of f parallel to level sets of g

i.e. where $\nabla f = \lambda \nabla g$

Example $\nabla f = \langle \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x, \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y \rangle = \lambda \nabla g = \lambda \langle 1, 1 \rangle$ $(\lambda \neq 0)$

$$\left.\begin{aligned} \frac{x}{\sqrt{x^2+y^2}} &= \lambda \\ \frac{y}{\sqrt{x^2+y^2}} &= \lambda \end{aligned}\right\} \quad \frac{x}{y} = 1 \Rightarrow x=y$$

now plug in to $g(x,y)=0$

$x+y-4=0$

$x+x-4=0 \quad \begin{aligned} x&=2 \\ y&=2 \end{aligned} \quad f(2,2) = 2\sqrt{2}$

Summary want to find max/min of f given $g=0$

solve $\nabla f = \lambda \nabla g$, $g=0$

Example find the point on the plane $x+2y+3z=4$ closest to the origin.

i.e. minimize $\underbrace{x^2+y^2+z^2}_{f(x,y,z)}$ subject to $\underbrace{x+2y+3z=4}_{g(x,y,z)}$.