Example $f(x,y,z) = xy + z^3$ $\qquad$ $x = s+t$, $y = s-t$, $z = st$

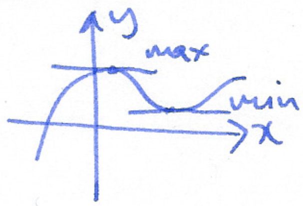
$f: \mathbb{R}^3 \to \mathbb{R}$ $\qquad$ $g: \mathbb{R}^2 \to \mathbb{R}^3$, $(s,t) \mapsto (x,y,z)$

so $f(g(s,t))$ makes sense.

Q: find $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial z}\cdot\dfrac{\partial z}{\partial s}$

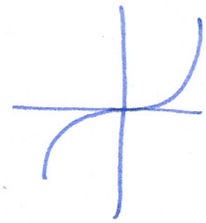
§14.7 optimization

recall: 1 var:



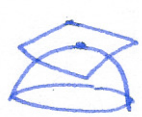
max, min $\Rightarrow \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = 0 \not\Rightarrow$ max, min

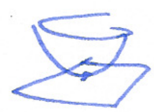


$y = x^3$.

2 vars local max



local min



$\Rightarrow$ horizontal tangent plane $z = \text{const}$.

recall: tangent plane to $z = f(x,y)$ at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

flat tangent plane $\Rightarrow$ $f_x(a,b) = 0$ and $f_y(a,b) = 0$

warning $f_x(a,b)$ and $f_y(a,b) = 0 \not\Rightarrow$ local max or min

Defn A critical point (a,b) is a point s.t. $\dfrac{\partial f}{\partial x}(a,b) = 0$ and $\dfrac{\partial f}{\partial y}(a,b) = 0$, or at least one of them does not exist.

Thm If $f(x,y)$ has a local max or min at (a,b) then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Example finding critical points $f(x,y) = x^2 - 2xy + 2y^2 + 3y + 1$

$\dfrac{\partial f}{\partial x} = 2x - 2y$ ①

$\dfrac{\partial f}{\partial y} = -2x + 4y + 3$ ②

① + ②: $2y + 3 = 0$ $\quad y = -3/2 \quad x = -3/2$

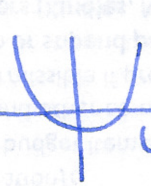
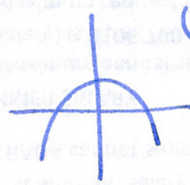
only one critical point $(-3/2, -3/2)$

Types of critical point: local max  local min  saddle  more complicated monkey saddle

Level sets    

Q: how to tell which are? A: Look at 2nd order / quadratic approx.

1 var $f(x) \approx f(a) + \underbrace{f'(a)}_{=0 \text{ if critical point}}(x-a) + \frac{1}{2} f''(a)(x-a)^2 + O(x^3)$

$f''(a) > 0$  local min $\qquad f''(a) < 0$  local max $\qquad f''(a) = 0$ no information

2 vars: Thm (2nd derivative test) Let (a,b) be a critical point for $f(x,y)$ and let $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$ then

1) if $D > 0$ then (a,b) is a minimum if $f_{xx}(a,b) > 0$
maximum if $f_{xx}(a,b) < 0$

2) if $D < 0$ then (a,b) is a saddle

3) if $D = 0$ no information.

why does this work?

Quadratic approximation is $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \begin{bmatrix} x-a & y-b \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$

quadratic terms are: $f_{xx}(a,b)(x-a)^2 + 2 f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2$

this is a quadratic function like x^2+y^2  $-x^2-y^2$  x^2-y^2 

up to change of coordinates, this looks like

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \leftarrow \lambda_1 x^2 + \lambda_2 y^2$

$\lambda_1, \lambda_2 > 0$ max
$\lambda_1, \lambda_2 < 0$ min
different sign saddle
either are 0 no information

Example $f(x,y) = (x^2+y^2)e^{-2x}$