- if $\underline{v}$ is not a unit vector, the directional derivative in direction $\underline{v}$ is $\frac{1}{\|\underline{v}\|} \nabla f \cdot \underline{v}$

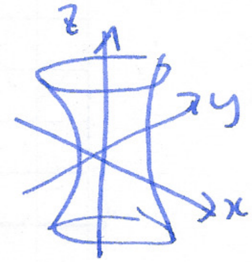
Applications

- <u>Finding normal vectors</u> sphere $x^2+y^2+z^2=r^2$

consider $f(x,y,z) = x^2+y^2+z^2$

$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ is normal vector

- <u>Finding tangent planes</u> hyperboloid $x^2+y^2=z^2+1$



find normal vectors: consider $f(x,y,z) = x^2+y^2-z^2$

then $\nabla f = \langle 2x, 2y, -2z \rangle$ so normal vector at $(1,1,1)$ is $\langle 2,2,-2 \rangle$

so tangent plane is $\underline{n} \cdot (\underline{x} - \underline{p}) = 0$ $\quad \langle 2,2,-2 \rangle \cdot (\langle x,y,z \rangle - \langle 1,1,1 \rangle) = 0$

$$2x + 2y - 2z = 2 .$$

§14.6 Chain rule

<u>recall</u> $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$ $\qquad (f(g(x)))' = f'(g(x)) \cdot g'(x)$.

$x \mapsto g(x) \mapsto f(g(x))$

general functions $f: \mathbb{R}^a \to \mathbb{R}^b$ $\quad f(x_1, \dots, x_a) = (f_1(x_1 \dots x_a), f_2(x_1 \dots x_a), \dots, f_b(x_1 \dots x_a))$

derivative $Df: \mathbb{R}^a \to \mathbb{R}^b$ is a linear map given by the matrix of partial derivatives:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_a} \\ \frac{\partial f_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial f_b}{\partial x_1} & & \cdots & \frac{\partial f_b}{\partial x_a} \end{bmatrix}$$

$b \times a$ matrix.

<u>Examples</u> $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x_1, x_2, x_3)$ $\quad Df = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right] = \nabla f$

$f: \mathbb{R} \to \mathbb{R}^3$ $\quad x_1 \mapsto (f_1(x_1), f_2(x_1), f_3(x_1))$ $\quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_3}{\partial x_1} \end{bmatrix} = \underline{f}'(t)$

<u>Composition</u>: $\mathbb{R}^a \xrightarrow{g} \mathbb{R}^b \xrightarrow{f} \mathbb{R}^c$ $\quad f(g(\underline{x}))$

$\underline{x} \longmapsto g(\underline{x}) \longmapsto f(g(\underline{x}))$

$Dg \qquad Df$

$$\underset{c\times a}{D(f(g(\underline{x})))} = \underset{c\times b}{Df(g(\underline{x}))}.\underset{b\times a}{Dg(\underline{x})}$$

<u>Example</u> ① $\mathbb{R} \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

$$x \longmapsto (g_1(x), g_2(x), g_3(x)) \longmapsto f(g_1(x), g_2(x), g_3(x))$$

$$\underset{1\times 1}{D(f(g(x)))} = \underset{1\times 3}{Df(g(\underline{x}))}\ \underset{3\times 1}{Dg(x)} = \nabla f(x) . g'(x) =$$

$$= \left< \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right> . \left< g_1'(x), g_2'(x), g_3'(x) \right>$$

② $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$

$(x_1, x_2) \quad (y_1, y_2)$

$$g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))$$

$$f(y_1, y_2) = (f_1(y_1, y_2), f_2(y_1, y_2))$$

$$D(f(g(\underline{x}))) = Df(g(\underline{x})). Dg(\underline{x})$$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad Dg = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{bmatrix}$$

$$Df\, Dg = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{\partial y_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial y_2} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

<u>mnemonic</u>: $f(x_1, \dots, x_n) \quad x_i(y_1, \dots, y_m)$

to find $$\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_i} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i}$$

"differentiate f wrt to all variables, mult. by $\frac{\partial x_j}{\partial y_i}$ and add".