Defn $f(x,y)$ is differentiable at (a,b) if

- $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist
- $f(x,y)$ is locally linear at (a,b)

in this case, the tangent plane is $z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Thm If $f_x(x,y)$ and $f_y(x,y)$ exist and are cts close to (a,b), then $f(x,y)$ is differentiable at (a,b).

Bad example $z^2 = x^2+y^2$ not differentiable at $(0,0)$

$z = \sqrt{x^2+y^2}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta \\ \frac{\partial f}{\partial y} &= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta \end{aligned}\right\} \text{not cts at } (0,0)!$$

§14.5 Gradient

2d: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y)$

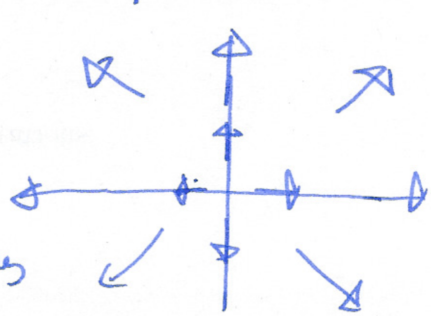
the **gradient** of f is $\underline{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$$\underline{\nabla} f(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle$$

3d: $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z)$ $\quad \underline{\nabla} f(a,b,c) = \left\langle \frac{\partial f}{\partial x}(a,b,c), \frac{\partial f}{\partial y}(a,b,c), \frac{\partial f}{\partial z}(a,b,c) \right\rangle$

note: $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$ (point $\mapsto$ vector) $\qquad \nabla f: \mathbb{R}^3 \to \mathbb{R}^3$ (point vector)

key facts:
- the gradient vector points in the direction of fastest rate of increase.
- $\|\underline{\nabla} f\|$ = fastest rate of increase.



Example $f(x,y) = x^2+y^2$ $\quad \nabla f = \langle 2x, 2y \rangle$

observation $\underline{\nabla} f$ perpendicular to level sets/contour lines

2d:  3d: 

useful properties

1) $\nabla(f+g) = \nabla f + \nabla g$

2) $\nabla(cf) = c\nabla f$ c constant

3) product rule: $\nabla(fg) = f\nabla g + g\nabla f$

4) chain rule: $\nabla(F(f(x,y,z))) = F'(f(x,y,z))\nabla f$ $F(t)$ differentiable

check this makes sense: $F \circ f$ $\mathbb{R}^3 \xrightarrow{f} \mathbb{R} \xrightarrow{F} \mathbb{R}$

$$\nabla(F(f(\underline{x}))) = \left< \frac{\partial}{\partial x} F(f(\underline{x})), \frac{\partial}{\partial y} F(f(\underline{x})), \frac{\partial}{\partial x} F(f(\underline{x})) \right>$$

$$= \left< F'(f(\underline{x})).\frac{\partial f}{\partial x}, F'(f(\underline{x})).\frac{\partial f}{\partial y}, F'(f(\underline{x})).\frac{\partial f}{\partial z} \right>$$

$$= F'(f(\underline{x})) \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right> = F'(f(\underline{x})).\nabla f.$$

Chain rule for paths

$f: \mathbb{R}^3 \to \mathbb{R}$ path $\underline{c}: \mathbb{R} \to \mathbb{R}^3$

$t \mapsto (x(t), y(t), z(t))$

composition: $c \circ f$ $\mathbb{R} \xrightarrow{\underline{c}} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

Thm $\frac{d}{dt}(f(\underline{c}(t))) = \nabla f(\underline{c}(t)).\underline{c}'(t)$

in $\mathbb{R}^2$: $\frac{d}{dt}(f(\underline{c}(t))) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>.\left< x'(t), y'(t) \right> = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Example temperature $T(x,y,z) = \frac{1}{x^2+y^2+z^2}$

move along an ellipse: $\underline{c}(t) = (5\cos t + 3, 4\sin t, 0)$

$$T(\underline{c}(t)) = \frac{1}{(5\cos t+3)^2 + (4\sin t)^2 + (0)^2}$$

$$\frac{d}{dt}(T(\underline{c}(t))) = \nabla T.\underline{c}'(t) = \left< \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right>.\left< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right>$$

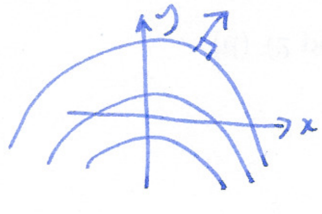
$$\langle -(x^2+y^2+z^2)^{-2}\cdot 2x,\ -(x^2+y^2+z^2)^{-2}\cdot 2y,\ -(x^2+y^2+z^2)^{-2}\cdot 2z\rangle \cdot \langle -5\sin t, 4\cos t, 0\rangle$$

$$= \frac{-2}{x^2+y^2+z^2}\langle x,y,z\rangle\cdot\langle -5\sin t, 4\cos t, 0\rangle$$

$$= \frac{-2}{(5\cos t+3)^2+(4\sin t)^2}\langle 5\cos t+3, 4\sin t, 0\rangle\cdot\langle -5\sin t, 4\cos t, 0\rangle$$

Directional derivatives

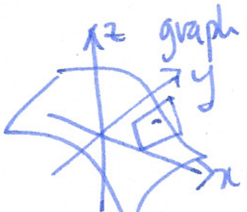
$f:\mathbb{R}^2\to\mathbb{R}$



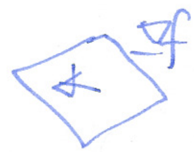
$\nabla f \perp$ level sets.

- rate of change in direction $\underline{\nabla f} = \|\nabla f\|$.
- rate of change in $\perp$ direction $= 0$

Q: what about rate of change in some other direction $\underline{v}$? (assume unit vector $\|\underline{v}\|=1$)



tangent plane at a point



Q: what is normal vector to graph?

$F(x,y,z) = f(x,y) - z$.

$\nabla F = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\rangle$.

Thm the directional derivative $D_{\underline{v}} f$ is:

$$\boxed{D_{\underline{v}} f(a,b) = \nabla f(a,b)\cdot\underline{v} \quad \text{if } \|\underline{v}\|=1}$$

Proof (sketch)

tangent plane has equation

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

so if $\underline{v} = \langle v_1, v_2\rangle$ then rate of change is $\frac{\partial f}{\partial x}(a,b)v_1 + \frac{\partial f}{\partial y}(a,b)v_2$

$= \nabla f(a,b)\cdot\underline{v}$

useful properties

- if $\underline{v}$ not a unit vector, define $D_{\underline{v}} f(a,b) = \underline{\nabla f}\cdot\underline{v}$
- $D_{\lambda\underline{v}}(f(a,b)) = \lambda \underline{\nabla f}\cdot\underline{v}$