easier: use polar coords $x = r\cos\theta$, $y = r\sin\theta$

$$\left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right)^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

Warning: $\lim\limits_{(x,0)\to(0,0)} f(x,y)$ exists $\not\Rightarrow$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists

Thm Limit laws (assume $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ exists)

then:

sums: $\lim\limits_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$

constant multiple: $\lim\limits_{(x,y)\to(a,b)} kf(x,y) = k\lim\limits_{(x,y)\to(a,b)} f(x,y)$

products: $\lim\limits_{(x,y)\to(a,b)} f(x,y)\,g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) \lim\limits_{(x,y)\to(a,b)} g(x,y)$

quotients: $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)}$ as long as denominator $\neq 0$.

Defn continuity $f(x,y)$ is cts at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Q: when is $f(x,y)$ cts? A: hard to know.

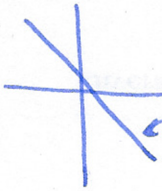
however: Thm "compositions of cts functions are cts".

so if f,g cts then fg, $f\circ g$ cts.
$f\pm g$
f/g

Example $f(x,y) = \dfrac{x-y}{x+y}$

note: $f(x,y) = x$ cts, $f(x,y) = y$ cts $\Big\}$ linear functions are cts.

so $x-y$ cts, $x+y$ cts, $\dfrac{x-y}{x+y}$ cts as long as $x+y \neq 0$ (i.e. $x \neq -y$)



← might not be cts here.

How to show a limit does not exist: $f(x,y) = \dfrac{x^2}{x^2+y^2}$ at $(0,0)$

find two different ways to get to $(0,0)$ which have different limits.

$$\left.\begin{aligned} \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+y^2} &= \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1 \\ \lim_{(0,y)\to(0,0)} \frac{x^2}{x^2+y^2} &= \lim_{(0,y)\to(0,0)} 0 = 0 \end{aligned}\right\} \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \text{ DNE.}$$

similar def$^{\text{n}}$s for functions of 3 or more vars.

Thm "compositions of cts functions are cts".

$f(x,y,z) = \dfrac{1}{x^2+y^2-z}$ cts where $x^2+y-z \neq 0$.

§14.3 Partial derivatives

$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto f(x,y)$

$\dfrac{\partial f}{\partial x}(x,y) = f_x =$ "diff. wrt x keeping y constant"

$\dfrac{\partial f}{\partial y}(x,y) = f_y =$ "diff wrt y keeping x constant"

Example $f(x,y) = x^2+y^2$ $\quad \dfrac{\partial f}{\partial x} = 2x \quad \dfrac{\partial f}{\partial y} = 2y$

$f(x,y) = xy \quad \dfrac{\partial f}{\partial x} = y \quad \dfrac{\partial f}{\partial y} = x$

$f(x,y) = xye^x + x^2y \quad f_x = ye^x + xye^x + 2xy \quad f_y = xe^x + x^2$.

<u>note</u>: f_x, f_y still functions of two variables, so we can partially differentiate again.

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$ $\quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ $\quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$ $\quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$

<u>Example</u>

$$f_{xx} = ye^x + xye^x + ye^x + 2y$$

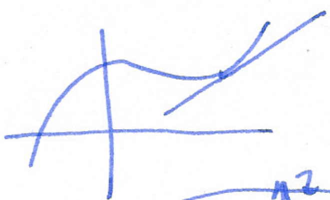
$$\left.\begin{aligned} f_{xy} &= e^x + xe^x + 2x \\ f_{yx} &= xe^x + e^x + 2x \end{aligned}\right\} \text{ equal !}$$

$$f_{yy} = 0$$

<u>Thm</u> If f_{xy} and f_{yx} are both cts, then $f_{xy} = f_{yx}$, "mixed partials are equal"

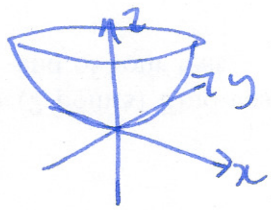
<u>Q</u>: what does this mean?

<u>1d</u>: $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x)$



$\frac{\partial f}{\partial x} = \frac{df}{dx}$ = rate of change of f wrt x / slope of tangent line.

<u>2d</u>: $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto f(x,y)$

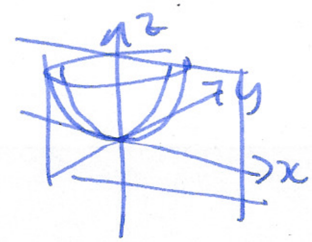


<u>Example</u> $f(x,y) = x^2 + y^2$

$f_x = 2x$

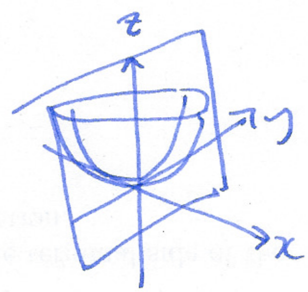
$f_y = 2y$

f_x = differentiate wrt x, <u>keeping y fixed</u>, i.e. $y = c$



$\leftrightarrow$ vertical slice parallel to xz-plane in $y = c$ $\quad f(x,c) = x^2 + c^2$ (i.e. parabola)

$\frac{\partial f}{\partial x}$ = slope in x-direction $\quad \frac{\partial f}{\partial x}(x,y)$!



f_y = differentiate wrt y <u>keeping x fixed</u>, i.e. $x = c$

$\leftrightarrow$ vertical slice parallel to yz-plane in $x = c$ $\quad f(c,y) = c^2 + y^2$

$\frac{\partial f}{\partial y}$ = slope in y-direction $\quad \frac{\partial f}{\partial y}(x,y)$!

f_{xx} = "2nd derivative in x-direction"

f_{yy} = "2nd derivative in y-direction"