§13.3 Arc length and speed

Recall: arc length of curves in $\mathbb{R}^2$

$(x(t), y(t))$

parameterized curve $(x(t), y(t))$ $t \in [a,b]$

length $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b \|\underline{r}'(t)\|\, dt$

this generalizes to parameterized curves in $\mathbb{R}^3$: $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$

length $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_a^b \|\underline{r}'(t)\|\, dt$

Example find the arc length of $\underline{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$ for $t \in [0, 2\pi]$

$\underline{r}'(t) = \langle -2\sin 2t, 2\cos 2t, 1 \rangle$

$\|\underline{r}'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 1} = \sqrt{5}$

$L = \int_0^{2\pi} \sqrt{5}\, dt = \left[\sqrt{5}\, t\right]_0^{2\pi} = 2\pi\sqrt{5}$.

observation $\underline{r}'(t)$ is tangent to the curve (as long as $\|\underline{r}'(t)\| \neq 0 \Leftrightarrow \underline{r}'(t) \neq \underline{0}$)

$\|\underline{r}'(t)\|$ is the speed at time t.

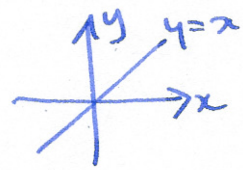
Example If a particle moves with position given by $\underline{r}(t) = \langle e^{2t}, t^{1/3}, \tan t \rangle$

find the speed at $t = 2$: $\underline{r}'(t) = \langle 2e^{2t}, \frac{1}{3}t^{-2/3}, \sec^2 t \rangle$

$\|\underline{r}'(t)\| = \| \langle 2e^4, \frac{1}{3 \cdot 2^{2/3}}, \sec^2 2 \rangle \| = \sqrt{4e^8 + \frac{1}{9 \cdot 2^{4/3}} + \sec^4 2}$.

Arc length parameterizations

problem: parameterizations not unique



$\underline{r}(t) = \langle t, t \rangle$
$\underline{r}(t) = \langle t^3, t^3 \rangle$

special parameterizations: arc length or unit speed parameterizations

Def$^{\underline{n}}$ $\underline{r}(t)$ is an arc length parameterization if $\|\underline{r}'(t)\| = 1$ for all t.

(i.e. you move along the curve with unit speed)

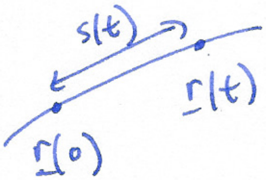
Example find an arc length parameterization for $\underline{r}(t) = \langle 2t, 1-2t, t \rangle$

① find arc length at time t: $s(t) = \int_0^t \|\underline{r}'(u)\|\, du$

$\underline{r}'(t) = \langle 2, -2, 1 \rangle$
$\|\underline{r}'(t)\| = \sqrt{4+4+1} = 3$

so $\quad s(t) = \int_0^t 3\,du = \left[3u\right]_0^t = 3t$

(2) find the inverse function for arc length



instead of $\underline{r}(t)$ want $\underline{r}(s^{-1}(t))$

(why: arc length of $\underline{r}(s^{-1}(t))$ is $s(s^{-1}(t)) = t$)

Example: $s^{-1}(t) = t/3$.

(3) write down reparameterized curve: $\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t)) = \langle \frac{2}{3}t, 1-\frac{2}{3}t, \frac{1}{3}t \rangle$

__check__: $\|\underline{r}(t)\| = \| \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle \| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$

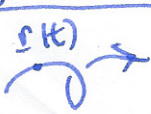
__summary__: $\underline{r}(t)$ arbitrary parameterization.

$s(t)$ arc length $\int_0^t \|\underline{r}'(t)\|\,dt$

$s^{-1}(t)$ inverse function of s

then the arc length parameterization is $\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t))$

§13.5 Motion in $\mathbb{R}^3$.



location/position $\underline{r}(t)$

velocity $\underline{v}(t) = \underline{r}'(t)$ $\quad$ speed $\|\underline{r}'(t)\| = \|\underline{v}(t)\|$

acceleration $\underline{a}(t) = \underline{r}''(t)$

__Example__ suppose position given by $\underline{r}(t) = \langle e^{-t}, \ln(t), \sqrt{t} \rangle$

then velocity $\underline{r}'(t) = \langle -e^{-t}, \frac{1}{t}, \frac{1}{2}t^{-1/2} \rangle$

acceleration $\underline{r}''(t) = \langle e^{-t}, -t^{-2}, -\frac{1}{4}t^{-3/2} \rangle$

__observation__: can reverse this, i.e. if we know $a(t)$

then $\underline{v}(t) = \int_0^t \underline{a}(u)\,du + \underline{v}_0$ $\quad$ $\underline{v}_0$ = initial velocity at $t=0$

$\underline{r}(t) = \int_0^t \underline{v}(u)\,du + \underline{r}_0$ $\quad$ $\underline{r}_0$ = initial position at $t=0$

__Example__ Find $\underline{r}(t)$ if $\underline{a}(t) = \langle 1, t \rangle$ $\quad \underline{v}_0 = \langle 1, 1 \rangle$, $\underline{r}_0 = \langle 2, 3 \rangle$

$\underline{v}(t) = \int_0^t \langle 1, u \rangle\,du + \underline{v}_0 = \langle \int_0^t du, \int_0^t u\,du \rangle + \langle 1, 1 \rangle$

$= \langle [u]_0^t, [\tfrac{1}{2}u^2]_0^t \rangle + \langle 1,1 \rangle = \langle t, \tfrac{1}{2}t^2 \rangle + \langle 1,1 \rangle = \langle t+1, \tfrac{1}{2}t^2+1 \rangle$

$\underline{r}(t) = \int_0^t \langle u+1, \tfrac{1}{2}u^2+1 \rangle \, du + \langle 2,3 \rangle$

$= \langle [\tfrac{1}{2}u^2+u]_0^t, [\tfrac{1}{6}u^3+u]_0^t \rangle + \langle 2,3 \rangle$

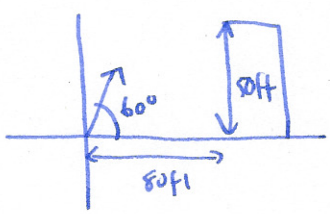
$= \langle \tfrac{1}{2}t^2+t+2, \tfrac{1}{6}t^3+t+3 \rangle.$

<u>Newton's laws of motion</u> $F = ma$ vector form $\underline{F} = m\underline{a}$

<u>note</u> this works if $\underline{F}$ not constant, say depends on position $\underline{F}(\underline{r}(t))$

then $\underline{F}(\underline{r}(t)) = m\,\underline{r}''(t)$.

<u>Example</u>



an object is thrown at an angle of 60° from the ground, want to hit something 50ft height 80ft away. how fast do you need to throw it?

gravity: downward force of magnitude $g = 32\ ft/s^2$, $9.8\ m/s^2$ $(10\ m/s^2)$

in vector form $\underline{F} = \langle 0, -gm \rangle$ (or $\langle 0, 0, -gm \rangle$)

$\underline{F} = m\underline{r}''(t) \Rightarrow \underline{r}''(t) = \langle 0, -g \rangle = \langle 0, -32 \rangle$

$\underline{r}'(t) = \langle 0, -32t \rangle + \underline{v_0}$

$\underline{r}(t) = \langle 0, -16t^2 \rangle + \underline{v_0}t + \underline{r_0}$

$\underline{r_0} = \underline{0}$ $\quad \underline{v_0} = v_0 \langle \cos 60°, \sin 60° \rangle = v_0 \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$.

$\underline{r}(t) = \langle 0, -16t^2 \rangle + tv_0 \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$ set $= \langle 80, 50 \rangle$ and solve for t.

$80 = \tfrac{1}{2}tv_0 \Rightarrow t = \dfrac{160}{v_0}$

$50 = -16t^2 + \tfrac{\sqrt{3}}{2}tv_0 \Rightarrow 50 = -16\left(\dfrac{160}{v_0}\right)^2 + \dfrac{\sqrt{3}}{2}\,\dfrac{160}{v_0}\,v_0$

$v_0^2 = \dfrac{16 \cdot (160)^2}{80\sqrt{3} - 50} \approx 4600\ ft/s.$

§14.1 Functions of many variables

Examples height above sea level $h(x,y)$
temperature $t(x,y,z)$

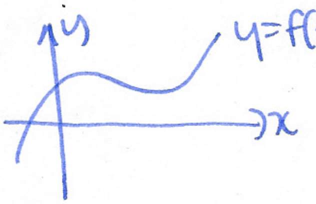
Defn A function of many variables is a function $f: \mathbb{R}^n \to \mathbb{R}$
domain range
(a $D \subset \mathbb{R}^n$).

Examples

$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto \sqrt{9-x^2-y^2}$ Q: what is D?

$f: \mathbb{R}^3 \to \mathbb{R}$
$(x,y,z) \mapsto x\sqrt{y} + \ln(z-1)$ Q: what is D?

Recall a function of one variable has a graph $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x)$

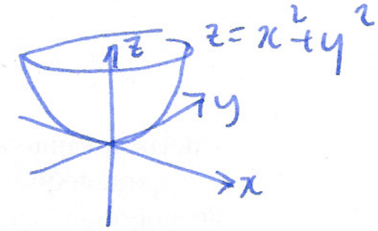


graph is $(x, f(x))$ in $\mathbb{R}^2$

similarly we can draw the graph of a function of two variables

$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto f(x,y)$

graph is $(x,y,f(x,y))$
i.e. a surface $z = f(x,y)$ is $\mathbb{R}^3$.



Example ① draw graph of $f(x,y) = x^2+y^2$

traces: horizontal : solve $f(x,y) = x^2+y^2 = c$ (circles of radius $\sqrt{c}$)

vertical : fix $x=c$ or $y=c$
$f(x,c) = x^2+c^2$ (parabolas)
$f(c,y) = c^2+y^2$ (parabolas)

② draw graph of $f(x,y) = x^2-y^2$

vertical traces are : $f(x,c) = x^2-c^2$ $\cup$
$f(c,y) = c^2-y^2$ $\cap$

③ linear functions $f(x,y) = ax+by+c$ graph is plane $z = ax+by+c$

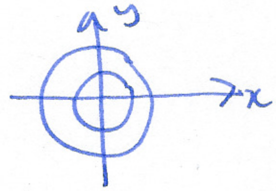
horizontal traces are straight lines.
vertical traces " "

Contour maps

The horizontal traces are known as contour lines or level sets

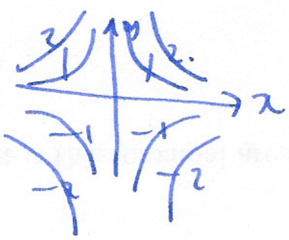
($\Leftrightarrow$ solutions to $f(x,y)=c$) contour lines/level sets contained in the domain.

Example $f(x,y) = \sqrt{4-x^2-y^2}$

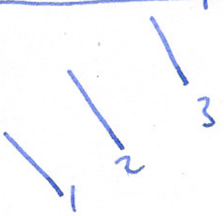


② $f(x,y) = x^2 y$

$f(x,y) = c \Leftrightarrow x^2 y = c \quad y = \frac{c}{x^2}$



On the interpretation of contours



far apart

$\updownarrow$

slow change

close together

$\updownarrow$

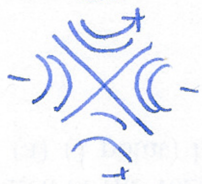
fast change

direction of fastest change is $\perp$ to contour line.

$\parallel$ to contour stays the same.

special patterns :

local max/min depending on labels.



saddle.

average rate of change $\frac{\Delta f}{\Delta \underline{r}}$

Function of 3 vars

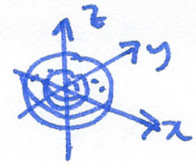
$f: \mathbb{R}^3 \to \mathbb{R}$

$(x,y,z) \mapsto f(x,y,z)$

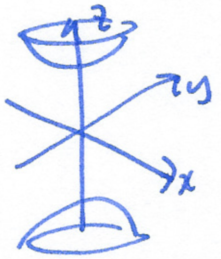
graphs : live in $\mathbb{R}^4$, hard to draw

level sets : $f(x,y,z) = c$, surfaces in $\mathbb{R}^3$, can draw these.

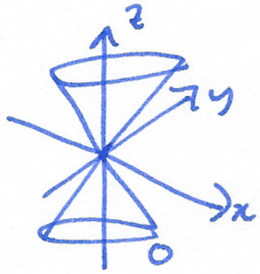
Examples ① $f(x,y,z) = x^2+y^2+z^2$



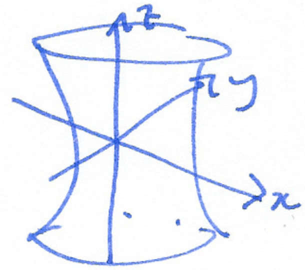
$f(x,y,z) = x^2+y^2-z^2$



$c<0$



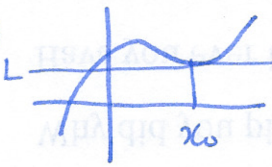
$f(x,y,z)=c=0$



$c>0$

§14.2 Limits and continuity for functions of many variables

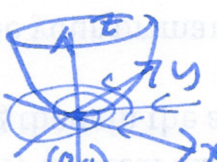
recall $y=f(x)$



$\lim_{x\to x_0} f(x) = L$ if $|f(x)-L|$ gets small as $|x-x_0|$ gets small.

only two directions, left and right.

2 vars $z=f(x,y)$



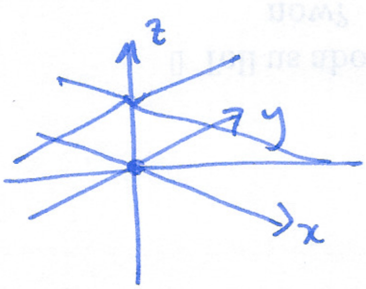
many ways to get to $(0,0)$

Def^n $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $|f(x,y)-L|$ gets small as $|(x,y)-(x_0,y_0)|$ gets small.

Bad example $f(x,y) = \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$

Q: what happens near $(0,0)$? $\lim_{(x,0)\to(0,0)} f(x,y) = \left(\frac{x^2}{x^2}\right)^2 = 1$

$\lim_{(0,y)\to(0,0)} f(x,y) = \left(\frac{-y^2}{y^2}\right)^2 = 1$



$\lim_{(t,t)\to(0,0)} f(x,y) = \left(\frac{0}{2t^2}\right)^2 = 0$