simple example in $\mathbb{R}^2$: $\underline{r}_1(t).\underline{r}_2(t) = \langle x_1(t), y_1(t)\rangle . \langle x_2(t), y_2(t)\rangle$

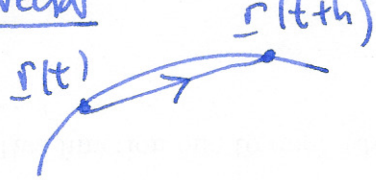
$$(\underline{r}_1(t).\underline{r}_2(t))' = (x_1x_2 + y_1y_2)' = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$$

$$= x_1'x_2 + y_1'y_2 + x_1x_2' + y_1y_2' = \langle x_1', y_1'\rangle . \langle x_2, y_2\rangle + \langle x_1, y_1\rangle . \langle x_2', y_2'\rangle$$

$$= \underline{r}_1'(t).\underline{r}_2(t) + \underline{r}_1(t).\underline{r}_2'(t). \quad \square .$$

<u>The derivative is the tangent vector</u> (or velocity vector)

$$r'(t) = \lim_{h \to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$$



<u>tangent line</u> : is given by $\underline{L}(t) = \underline{r}(t_0) + t\,\underline{r}'(t_0)$

<u>Example</u> find tangent line to helix $\underline{r}(t) = \langle \cos t, \sin t, t\rangle$ at $t=1$.

<u>Example</u> show $\underline{r}(t)$ and $\underline{r}'(t)$ are orthogonal if $\underline{r}(t)$ has unit length.

unit length $\|\underline{r}'(t)\| = 1$
$$\underline{r}(t).\underline{r}(t)$$

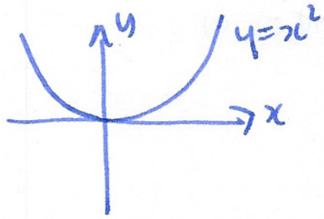
so $(\underline{r}(t).\underline{r}(t))' = \underline{r}'(t).\underline{r}(t) + \underline{r}(t).\underline{r}'(t) = 2\,\underline{r}'(t).\underline{r}(t) = 0$

$\Rightarrow \underline{r}(t) \perp \underline{r}'(t)$

<u>geometric interpretation:</u>

sphere $\|\underline{r}\| = 1$

<u>Example</u>



$\underline{r}(t) = \langle t, t^2\rangle$

$\underline{r}'(t) = \langle 1, 2t\rangle$ note $\|\underline{r}'(t)\| \neq 0$ in this parameterization

<u>Q</u> : $\underline{r}(t) = \langle t^3, t^6\rangle$?

<u>Integration</u>

<u>Defn</u> define integration componentwise, i.e. $\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$

$$\int_a^b \underline{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

the integral exists if each of the components is integrable.

__Example__ $\int_0^1 \langle t, e^{2t}, \frac{1}{1+x^2} \rangle dt = \langle \int_0^1 t\,dt, \int_0^1 e^{2t} dt, \int_0^1 \frac{1}{1+x^2} dt \rangle$

$\langle \left[\frac{1}{2}t^2\right]_0^1, \left[\frac{1}{2}e^{2t}\right]_0^1, \left[\tan^{-1}(x)\right]_0^1 \rangle = \langle \frac{1}{2}, \frac{1}{2}e^2 - \frac{1}{2}, \underbrace{\tan^{-1}(1)}_{\pi/4} \rangle$

Antiderivatives

__Def$^{\underline{n}}$__ An anti-derivative of $\underline{r}(t)$ is a function $\underline{R}(t)$ s.t. $\underline{R}'(t) = \underline{r}(t)$

__Th$^{\underline{m}}$ 4__ Let $\underline{R}_1(t)$ and $\underline{R}_2(t)$ be antiderivatives of $\underline{r}(t)$, i.e. $R_1'(t) = R_2'(t) = r(t)$.

Then $\underline{R}_1(t) = \underline{R}_2(t) + \underline{c}$ $\quad \underline{c}$ constant vector.

__General antiderivative__ $\quad \int \underline{r}(t)dt = \underline{R}(t) + \underline{c}$

__Fundamental theorem of calculus__ (vector valued version)

$\underline{r}(t)$ continuous on $[a,b]$ and $\underline{R}(t)$ an antiderivative of $\underline{r}(t)$

then $\int_a^b \underline{r}(t)dt = \underline{R}(b) - \underline{R}(a)$.

__Example__ A particle moves with velocity $\underline{r}(t) = \langle t, \sin t \rangle$. If it starts at $\langle 1, 1 \rangle$ at $t=0$, where is it at $t=4$?

$\int_0^4 \underline{r}(t)dt = \underline{R}(4) - \underline{R}(0) \qquad \underline{R}(t) = \langle \frac{1}{2}t^2, -\cos t \rangle$

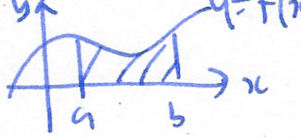
$= \langle 8, -\cos 4 \rangle - \langle 0, -1 \rangle = \langle 8, 1 - \cos 4 \rangle$

note

$$\begin{array}{ll} \text{position} & \underline{r}(t) \\ \downarrow \frac{d}{dt} & \uparrow \int \\ \text{velocity} & \underline{r}'(t) \\ \downarrow \frac{d}{dt} & \uparrow \int \\ \text{acceleration} & \underline{r}''(t) \end{array}$$

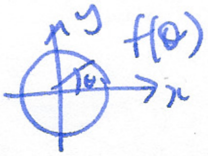
__Warning/motivation__

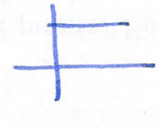
$f: \mathbb{R} \to \mathbb{R}$



$y = f(x)$ has integral $F(x) = \int_a^b f(x)dx$ = area under curve.

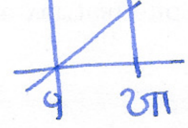
__note__ this doesn't work for $f: \underset{\text{circle}}{S^1} \to \mathbb{R}$





$f(\theta) = 1$

$f'(\theta) = 0$



$\int f(\theta)d\theta = \theta ? = F(\theta)$

__not__ periodic $\quad F(0) = 0$

$F(2\pi) = 2\pi$ #.