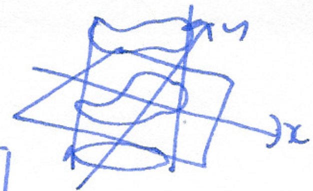
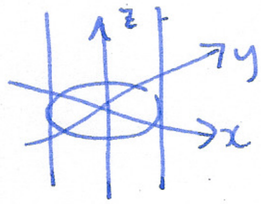
<u>cylinders</u> general cylinder : $C$ some curve in $xy$-plane :

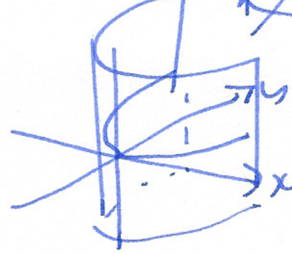
the cylinder over $C$ is all vertical lines through $C$.




Example $x^2+y^2=r^2$




$y=x^2$




## §13.1 Vector valued functions

real valued functions of 1-variable $\quad f: \mathbb{R} \longrightarrow \mathbb{R}$
$\qquad x \longmapsto f(x)$

parameterized curves / vector valued functions of 1-variable

$f: \mathbb{R} \longrightarrow \mathbb{R}^2$ $\qquad\qquad f: \mathbb{R} \longrightarrow \mathbb{R}^3$

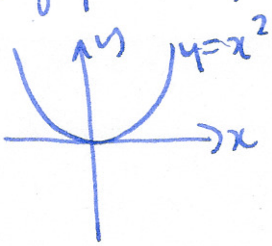
$t \longmapsto \langle f(t), g(t) \rangle$ $\qquad\qquad t \longmapsto \langle f(t), g(t), h(t) \rangle$

alternate notation

$t \longmapsto f(t)\,\underline{i} + g(t)\,\underline{j}$ $\qquad\qquad t \mapsto f(t)\,\underline{i} + g(t)\,\underline{j} + h(t)\,\underline{k}$

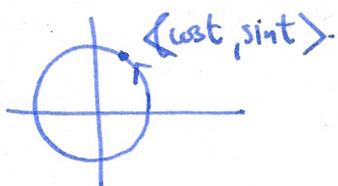
<u>Example</u>

① graph of a function of 1-variable




$x \longmapsto \langle x, x^2 \rangle$ (in general $x \longmapsto \langle x, f(x) \rangle$)

② $\underline{r}(t) = \langle \cos(t), \sin(t) \rangle$




<u>observation</u> $\cos^2 t + \sin^2 t = 1$
$x^2+y^2 = 1$.

<u>warning</u> : the same curve has many different parameterizations

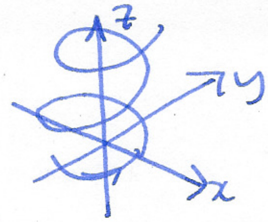
$\langle \cos t, \sin t \rangle$ $\quad 0 \leq t \leq 2\pi$

$\langle \cos(2t), \sin(2t) \rangle$ $\quad 0 \leq t \leq \pi$ (twice as fast)

<u>note</u> : $\langle \cos(t), \sin(t) \rangle$ $\quad 0 \leq t \leq \pi$ only goes half way round

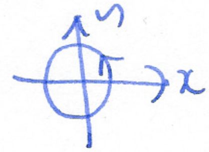
$\langle \cos(t), \sin(t) \rangle$ $\quad t \in \mathbb{R}$ goes round infinitely often.

**Example** $\underline{r}(t) = \langle \cos(t), \sin(t), t \rangle$
(helix)

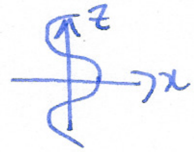



**projections**: set one coordinate equal to 0

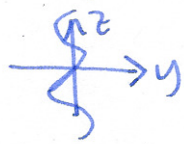
$\langle \cos t, \sin t, 0 \rangle$ projection onto $xy$-plane




$\langle \cos t, 0, t \rangle$ $xz$-plane

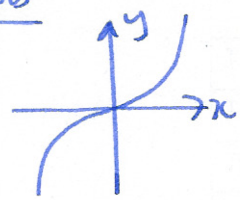



$\langle 0, \sin t, t \rangle$ $yz$-plane




## Finding parameterizations

**Example** ① $y = x^3$




$\underline{r}(t) = \langle t, t^3 \rangle$

② circle of radius 4 in $xz$-plane with center $(1,2,3)$

unit circle in $xy$-plane : $\underline{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$

$xz$-plane : $r(t) = \langle \cos t, 0, \sin t \rangle$

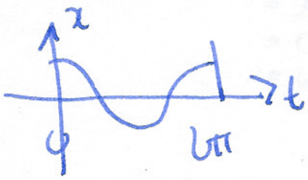
radius 4 : $r(t) = \langle 4\cos t, 0, 4\sin t \rangle$

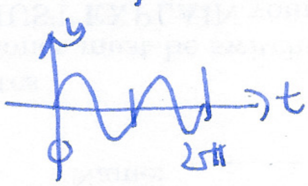
translate : $r(t) = \langle 4\cos t + 1, 2, 4\sin t + 3 \rangle$.

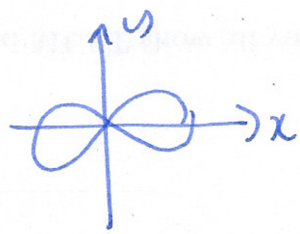
## Drawing parameterized curves

**Example** $\underline{r}(t) = \langle \cos(t), \sin(2t) \rangle$

try drawing $(x,t)$ $(y,t)$ pictures.




## §13.2 Calculus of vector valued functions

**limits**. **Defn** A vector valued function $\underline{r}(t)$ has a limit $\underline{v}$ at $t_0$ if $\lim_{t \to t_0} \|\underline{r}(t) - \underline{v}\| = 0$. If this happens we write $\lim_{t \to t_0} \underline{r}(t) = \underline{v}$

**Thm 1** Limits are computed componentwise

$\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$ has a limit as $t \to t_0 \iff$ each component function has a limit, and

$$\lim_{t\to t_0} \underline{r}(t) = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\right\rangle$$

**Proof** (sketch) $\|\underline{r}(t) - \underline{v}\|^2 = (x(t)-v_1)^2 + (y(t)-v_2)^2 + (z(t)-v_3)^2$

LHS $\to 0$ $\Rightarrow$ each term on RHS $\to 0$ $\square$.

**Example** find $\lim_{t\to 0} \underline{r}(t)$ where $\underline{r}(t) = \langle t^2, t, \frac{\sin t}{t}\rangle$

$$\lim_{t\to 0} \underline{r}(t) = \left\langle \lim_{t\to 0} t^2, \lim_{t\to 0} t, \lim_{t\to 0} \frac{\sin t}{t}\right\rangle = \langle 0,0,1\rangle$$

**Continuity**

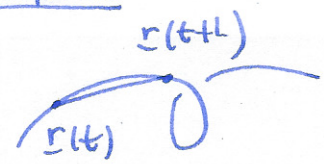
**Defn** $\underline{r}(t)$ is continuous $\iff$ all of its components are continuous

**Differentiation**

**Defn** The derivative of $\underline{r}(t)$ is $\underline{r}'(t) = \frac{d\underline{r}}{dt} = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$

if this limit exists, we say $\underline{r}(t)$ is differentiable

**Remark** $\frac{\underline{r}(t+h) - \underline{r}(t)}{h}$ makes sense!




**Thm 2** Derivatives are computed componentwise.

if $\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$ then $\underline{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$

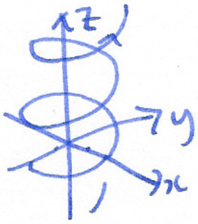
**Proof** (sketch) derivative defined as a limit, limits computed componentwise. $\square$.

**Example**
$\underline{r}(t) = \langle \cos t, \sin t, t\rangle$
$\underline{r}'(t) = \langle -\sin t, \cos t, 1\rangle$

$\underline{r}(t) = \langle e^{2t}, t\sin t, \frac{\cos t}{t}\rangle$
$\underline{r}'(t) = \langle 2e^{2t}, \sin t + t\cos t, \frac{t\cdot -\sin t - \cos t}{t^2}\rangle$

**Higher derivatives** $\underline{r}''(t) = \frac{d^2\underline{r}}{dt^2} = \lim_{h\to 0} \frac{\underline{r}'(t+h) - \underline{r}'(t)}{h}$

**Example** $\underline{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$.




if $\underline{r}(t)$ is position
$\underline{r}'(t)$ is velocity
$\underline{r}''(t)$ is acceleration.

**Rules for differentiation** (assume all functions differentiable)

- sums: $(\underline{r}_1(t)+\underline{r}_2(t))' = \underline{r}_1'(t)+\underline{r}_2'(t)$
- scalar multiplication $(c\,\underline{r}(t))' = c\,\underline{r}'(t)$ $c$ constant (does not depend on $t$)
- product rule: $f(t)$ scalar function (differentiable)
  $(f(t)\underline{r}(t))' = f(t)\underline{r}'(t) + f'(t)\underline{r}(t)$
- chain rule $\frac{d}{dt}(\underline{r}(g(t))) = g'(t)\,\underline{r}'(g(t))$

Warning: $g(\underline{r}(t))$ doesn't make sense!
$\underline{r}_1(\underline{r}_2(t))$ " " "

Proof (sketch) everything is defined componentwise.

product rule: $f(t)\,\underline{r}(t) = f(t)\langle x(t), y(t), z(t)\rangle = \langle f(t)x(t), f(t)y(t), f(t)z(t)\rangle$

$(f(t)\underline{r}(t))' = \langle f'x+fx', f'y+fy', f'z+fz'\rangle = f'\langle x,y,z\rangle + f\langle x',y',z'\rangle$

$= f'(t)\underline{r}(t) + f(t)\underline{r}'(t).$ $\square$.

**Thm 3 Product rule for dot and cross products**

dot product: $(\underline{r}_1(t)\cdot\underline{r}_2(t))' = \underline{r}_1(t)\cdot\underline{r}_2'(t) + \underline{r}_1'(t)\cdot\underline{r}_2(t)$

cross product: $(\underline{r}_1(t)\times\underline{r}_2(t))' = \underline{r}_1(t)\times\underline{r}_2'(t) + \underline{r}_1'(t)\times\underline{r}_2(t).$

Warning: order matters in cross product!

Proof (sketch) dot, cross products defined componentwise.