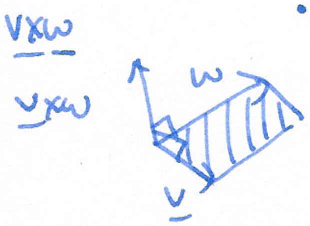
useful facts



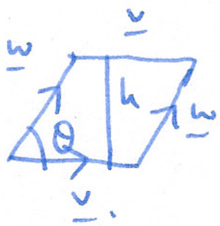
- $\|\underline{v} \times \underline{w}\|$ = area of parallelogram determined by $\underline{v}$ and $\underline{w}$

  area $A = \|\underline{v} \times \underline{w}\|$

- three vectors $\underline{u}, \underline{v}, \underline{w}$ determine a parallelepiped

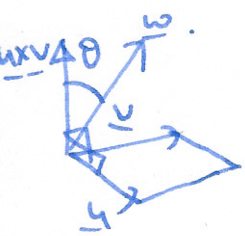
  volume $V = |\underline{u} \cdot (\underline{v} \times \underline{w})|$ (order doesn't matter)

check



$$\begin{aligned}\text{area} &= \text{base} \times \text{height} \\ &= \|\underline{v}\| h \\ &= \|\underline{v}\| \|\underline{w}\| \sin\theta = \|\underline{v} \times \underline{w}\|.\end{aligned}$$

volume of parallelepiped is = area of base × height



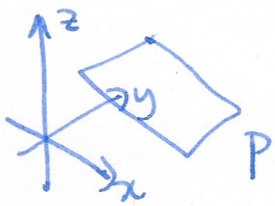
$$\begin{aligned}&= \big| \|\underline{u} \times \underline{v}\| \|\underline{w}\| \cos\theta \big| \\ &= |\underline{w} \cdot (\underline{u} \times \underline{v})|\end{aligned}$$

Notation: $\underline{u} \cdot (\underline{v} \times \underline{w})$ is called triple scalar product.

Warning: $\underline{u} \cdot (\underline{v} \times \underline{w})$ makes sense $\quad (\underline{u} \cdot \underline{v}) \times \underline{w}$ does not!

Note: if
$\underline{u} = \langle u_1, u_2, u_3 \rangle$
$\underline{v} = \langle v_1, v_2, v_3 \rangle$
$\underline{w} = \langle w_1, w_2, w_3 \rangle$
then
$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

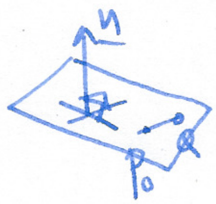
§12.5 Planes in $\mathbb{R}^3$



claim: a plane P in $\mathbb{R}^3$ is defined by a single linear equation

$ax+by+cz=d$

Proof:



let P_0 be a point on P $\quad P_0 = (x_0, y_0, z_0)$

let $\underline{n}$ be the normal vector to P $\quad \underline{n} = \langle a, b, c \rangle$

let Q be some other point on P $\quad Q = (x, y, z)$

then $\overrightarrow{P_0Q} \cdot \underline{n} = 0$ i.e. $\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax+by+cz = \underbrace{ax_0+by_0+cz_0}_{d}$$

summary $\quad ax+by+cz=d$ equation of a plane (scalar form)

$$\left.\begin{array}{l} \overrightarrow{P_0Q} \cdot \underline{n} = 0 \\ \text{or } (\underline{x}-\underline{p}) \cdot \underline{n} = 0 \end{array}\right\} \text{equation of a plane (vector form)}$$

Example find equation of plane through $P_0 = (1,2,3)$ with normal vector

$\underline{n} = \langle 2,1,-1 \rangle$: $\quad \underline{n} \cdot \overrightarrow{P_0Q} = 0 \quad \langle 2,1,-1 \rangle \cdot \langle x-1, y-2, z-3 \rangle = 0$

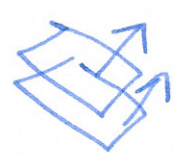
$$2x-2+y-2-z+3=0$$

$$2x+y-z=1$$

observations · $ax+by+cz=d$

has normal vector $\underline{n} = \langle a, b, c \rangle$

· parallel planes have the same normal vector



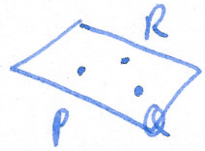
Example find the plane parallel to $x-3y+2z=4$

through the point $P = (6,4,2)$ $\quad \underline{n} = \langle 1,-3,2 \rangle$

$(\underline{x}-\underline{P}) \cdot \underline{n} = 0 \quad (x-6)-3(y-4)+2(z-2)=0$

$$x-3y+2z = 6-12+4 = -2$$

• three points determine a plane:



find normal vector $\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

<u>Example</u> $P = (1,0,1)$ $Q = (2,3,1)$ $R = (-1,-1,3)$

$\overrightarrow{PQ} = \langle 1,3,0\rangle$ $\overrightarrow{PR} = \langle -2,-1,2\rangle$

$$\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & \underline{k} \\ 1 & 3 & 0 \\ -2 & -1 & 2 \end{vmatrix} = i \begin{vmatrix} 3 & 0 \\ -1 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}$$

$$= \langle 6, -2, 5\rangle$$

equation: $6(x-1) - 2y + 5(z-1) = 0$

• find intersection of a plane and a line

plane: $2x - 3y + 4z = 2$

line: $\langle 1,2,1\rangle + t\langle -2,1,1\rangle \leftarrow \begin{cases} x = 1-2t \\ y = 2+t \\ z = 1+t \end{cases}$

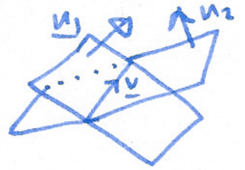
substitute in to equation of plane: $2(1-2t) - 3(2+t) + 4(1+t) = 2$

$2 - 4t - 6 - 3t + 4 + 4t = 2$

$-3t = 2$ $\quad t = -2/3$.

so point is $\langle 1+\frac{4}{3}, 2-\frac{2}{3}, 1-\frac{2}{3}\rangle = \langle \frac{7}{3}, \frac{4}{3}, \frac{1}{3}\rangle$. <u>check!</u>

• <u>intersection of two planes</u>



direction vector $\underline{v}$ for line is perpendicular to both $\underline{n}_1, \underline{n}_2$.
so $\underline{v} = \underline{n}_1 \times \underline{n}_2$, then just find a point on the line.

<u>Example</u> $x + y - z = 2$ $\quad \underline{n}_1 = \langle 1,1,-1\rangle$

$x + 2y + z = 4$ $\quad \underline{n}_2 = \langle 1,2,1\rangle$

$$\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \langle 3, -2, 1\rangle$$

find point on intersection

<u>try</u>: set $z = 0$: $\left.\begin{matrix} x + y = 2 \\ x + 2y = 4 \end{matrix}\right\}$ solve $\begin{matrix} x = 0 \\ y = 2 \end{matrix}$ so $(0,2,0)$ works

<u>eqn</u>: $(\underline{x} - \langle 0,2,0\rangle) . \langle 3,-2,1\rangle = 0$.