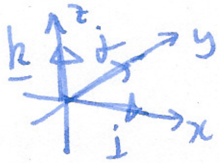
- **scalar multiplication** $\lambda \underline{v} = \lambda \langle v_1, v_2, v_3 \rangle = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$

- **standard basis vectors**



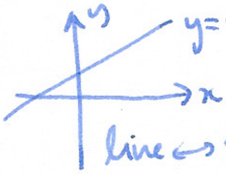
$\underline{i} = \langle 1,0,0 \rangle$
$\underline{j} = \langle 0,1,0 \rangle$
$\underline{k} = \langle 0,0,1 \rangle$

every vector is a linear combination of the standard basis vectors

$\underline{v} = \langle a,b,c \rangle = a\underline{i} + b\underline{j} + c\underline{k} = a\langle 1,0,0 \rangle + b\langle 0,1,0 \rangle + c\langle 0,0,1 \rangle$

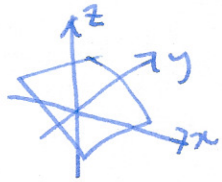
Equations of lines in $\mathbb{R}^3$

recall



$y = mx+b$

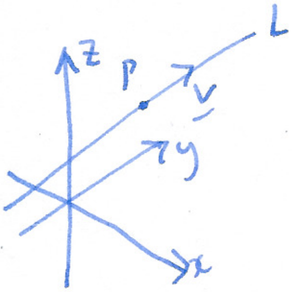
line $\leftrightarrow$ single linear equation



$ax+by+cz=d$

single linear equation $\leftrightarrow$ plane!

for lines in $\mathbb{R}^3$ we can write a <u>parametric equation</u> $\underline{r}(t)$



a line L is determined by a point P and a vector $\underline{v}$

any point on L can then be written as $\underline{r}(t) = \overrightarrow{OP} + t\underline{v}$

t is called the <u>parameter</u> if $\overrightarrow{OP} = \langle a,b,c \rangle$

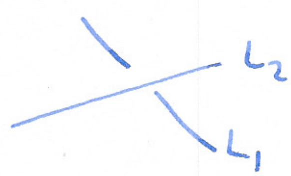
$\underline{v} = \langle v_1, v_2, v_3 \rangle$

then $\underline{r}(t) = \langle a,b,c \rangle + t\langle v_1, v_2, v_3 \rangle = \langle a+tv_1, b+tv_2, c+tv_2 \rangle$

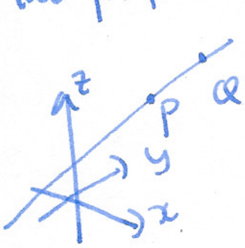
problem L has many different parameterizations.

$\mathbb{R}^2$: any two lines either intersect or are parallel

$\mathbb{R}^3$: lines may intersect, be parallel, or neither (skew)



Two points determine a line



$P = (p_1, p_2, p_3)$ then $\underline{v} = \overrightarrow{PQ} = \langle q_1-p_1, q_2-p_2, q_3-p_3 \rangle$

$Q = (q_1, q_2, q_3)$ so $\underline{r}(t) = \langle p_1, p_2, p_3 \rangle + t\langle q_1-p_1, q_2-p_2, q_3-p_3 \rangle$

§12.3 Dot product and angle

$\underline{v}, \underline{w}$ vectors

dot product $\underline{v}.\underline{w}$

geometric defn : $\|\underline{v}\| \, \|\underline{w}\| \cos\theta$ θ angle between vectors

coordinate defn : $\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\underline{w} = \langle w_1, w_2, w_3 \rangle$

$$\underline{v}.\underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Thm two defns equivalent.

useful properties

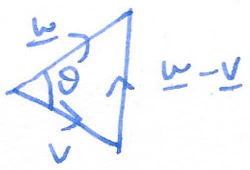
- $\underline{0}.\underline{v} = \underline{v}.\underline{0} = 0$
- $\underline{v}.\underline{w} = \underline{w}.\underline{v}$ (commutativity).
- $(\lambda \underline{v}).\underline{w} = \lambda(\underline{v}.\underline{w}) = \underline{v}.(\lambda \underline{w})$ (scalar distributivity)
- $\underline{u}.(\underline{v}+\underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}$ (distributivity)
- $\underline{v}.\underline{v} = \|\underline{v}\|^2$ (length)

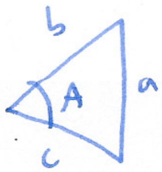
Angle between two vectors

$$\underline{v}.\underline{w} = \|\underline{v}\| \, \|\underline{w}\| \cos\theta$$

$$\cos\theta = \frac{\underline{v}.\underline{w}}{\|\underline{v}\| \, \|\underline{w}\|}$$

Proof (uses law of cosines)





$$a^2 = b^2 + c^2 - 2ab\cos A$$

$$\|\underline{w}-\underline{v}\|^2 = \|\underline{w}\|^2 + \|\underline{v}\|^2 - 2\|\underline{v}\| \, \|\underline{w}\| \cos\theta$$

$$\|\underline{w}-\underline{v}\|^2 = (\underline{w}-\underline{v}).(\underline{w}-\underline{v}) = \underline{w}.\underline{w} - \underline{w}.\underline{v} - \underline{v}.\underline{w} + \underline{v}.\underline{v}$$

$$= \|\underline{w}\|^2 - 2\underline{v}.\underline{w} + \|\underline{v}\|^2$$

$$\cancel{\|\underline{w}\|^2} - 2\underline{v}.\underline{w} + \cancel{\|\underline{v}\|^2} = \cancel{\|\underline{w}\|^2} + \cancel{\|\underline{v}\|^2} - 2\|\underline{v}\| \, \|\underline{w}\| \cos\theta$$

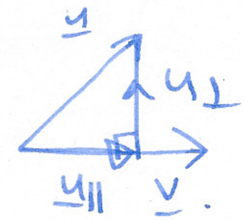
$$\cos\theta = \frac{\underline{v}.\underline{w}}{\|\underline{v}\| \, \|\underline{w}\|} \quad \square.$$

Def$^{\underline{n}}$ $\underline{u}, \underline{v}$ are perpendicular (or orthogonal) iff $\underline{u}\cdot\underline{v} = 0$ $(\underline{u} \perp \underline{v})$

Projections given vectors $\underline{u}, \underline{v}$ we can write $\underline{u}$ as $\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$

where $\underline{u}_{\parallel}$ is parallel to $\underline{v}$

$\underline{u}_{\perp}$ is perpendicular to $\underline{v}$



$\underline{u}_{\parallel}$ is called the projection of $\underline{u}$ to $\underline{v}$, also written $\text{proj}_{\underline{v}}(\underline{u})$

Q: what is $\underline{u}_{\parallel}$?

A: $\underline{u}_{\parallel} = c\underline{v}$ for some number c.

$$\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$$

$$\underline{u} = c\underline{v} + \underline{u}_{\perp}$$

take dot product with $\underline{v}$:

$$\underline{u}\cdot\underline{v} = c\,\underline{v}\cdot\underline{v} + \underbrace{\underline{u}_{\perp}\cdot\underline{v}}_{=0 \text{ as } \underline{u}_{\perp}, \underline{v} \text{ are perpendicular}}$$

$$\Rightarrow \quad c = \frac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}} \quad \text{so} \quad \underline{u}_{\parallel} = \text{proj}_{\underline{v}}(\underline{u}) = \frac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\,\underline{v}$$

summary given $\underline{u}, \underline{v}$ $\quad \underline{u} = \underbrace{\underline{u}_{\parallel}}_{\text{proj}_{\underline{v}}(\underline{u})} + \underline{u}_{\perp}$

where $\text{proj}_{\underline{v}}(\underline{u}) = \underline{u}_{\parallel} = \dfrac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\,\underline{v}$ and $\underline{u}_{\perp} = \underline{u} - \text{proj}_{\underline{v}}(\underline{u})$.

Example $\underline{u} = \langle 1,2,3\rangle$ $\quad \underline{u}_{\parallel} = \mathrm{proj}_{\underline{v}}(\underline{u}) = \dfrac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\underline{v} = \dfrac{6}{3}\langle 1,1,1\rangle$

$\underline{v} = \langle 1,1,1\rangle$ $\quad = \langle 2,2,2\rangle$

$$\underline{u}_{\perp} = \underline{u} - \mathrm{proj}_{\underline{v}}(\underline{u}) = \langle 1,2,3\rangle - \langle 2,2,2\rangle = \langle -1,0,1\rangle$$

check: $\underline{u}_{\perp}\cdot\underline{v} = 0$ $\quad \langle -1,0,1\rangle\cdot\langle 1,1,1\rangle = 0.$

§12.4 Cross product

dot product $\underline{v}\cdot\underline{w} \leftarrow$ scalar!

cross product $\underline{v}\times\underline{w} \Leftarrow$ vector!

recall A 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ has determinant $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

Example $\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 1\end{bmatrix}\right) = 2-1 = 1$

A 3x3 matrix: $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$

Defn if $\underline{v} = \langle v_1, v_2, v_3\rangle$ $\quad \underline{w} = \langle w_1, w_2, w_3\rangle$

then $\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix} = \underline{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3\end{vmatrix} - \underline{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3\end{vmatrix} + \underline{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2\end{vmatrix}$

$$= \langle v_2w_3 - v_3w_2,\ v_1w_3 - v_3w_1,\ v_1w_2 - w_1v_2\rangle$$

Example $\underline{v} = \langle 1,2,3\rangle$

$\underline{w} = \langle 1,-1,-1\rangle$

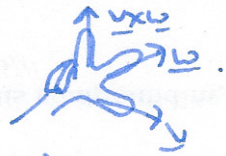
$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1\end{vmatrix} = \underline{i}\underbrace{\begin{vmatrix} 2 & 3 \\ -1 & -1\end{vmatrix}}_{-2+3} - \underline{j}\underbrace{\begin{vmatrix} 1 & 3 \\ 1 & -1\end{vmatrix}}_{-(-1-3)} + \underline{k}\underbrace{\begin{vmatrix} 1 & 2 \\ 1 & -1\end{vmatrix}}_{-1-2}$

$= \cancel{\langle 1,4,-3\rangle}$

$= \langle 1,4,-3\rangle$

Thm (Geometric defn of cross product)

$v \times w$ is the unique vector which is

1) perpendicular to both v and w
2) $\|v \times w\| = \|v\| \, \|w\| \, |\sin\theta|$ (θ angle between v and w)
3) $(v, w, v \times w)$ is a right handed triple .

warning : $v \times w = -w \times v$ (anti-commutative!)

note : $v \times v = 0 = -v \times v$.

Thm Useful properties

- $w \times v = -v \times w$
- $v \times v = 0$
- $v \times w = 0 \iff w = \lambda v$ for some $\lambda \in \mathbb{R}$.
- $(\lambda v) \times w = v \times (\lambda w) = \lambda (v \times w)$
- $(u + v) \times w = u \times w + v \times w$
- $u \times (v + w) = u \times v + u \times w$

special case

$$i \times j = k \qquad j \times i = -k$$
$$j \times k = i \qquad k \times j = -i$$
$$k \times i = j \qquad i \times k = -j$$

Alternate way of computing $v \times w$:

$v = \langle 1, 0, 1 \rangle = i + k$ then $v \times w = (i + k) \times (-j)$

$w = \langle 0, -1, 0 \rangle = -j$

$$= -i \times j - k \times j = -k + i .$$