Math 233 Calculus 3 Fall 18 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find and classify the critical points for the function $f(x, y) = e^y + xe^y + e^{-2x}$.

$f_x = e^y - 2e^{-2x} = 0$

$f_y = e^y + xe^y = 0$

$e^y(1+x) = 0 \Rightarrow x = -1$

$e^y = 2e^{+2}$

$y = \ln(2) + 2$.

critical point $(-1, \ln(2) - 2)$

$f_{xx} = 4e^{-2x}$

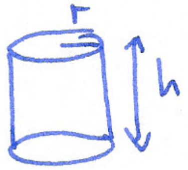
$f_{xy} = e^y$

$f_{yy} = e^y + xe^y$

$D = f_{xx} f_{yy} - f_{xy}^2$

$D(-1, \ln(2)+2) = (*).(0) - 4e^{+4} < 0 \Rightarrow$ saddle.

(2) (10 points) The UK postal website says that for cylinder-shaped packages, the length of the item plus twice the diameter of the flat end must not exceed 104cm. What is the volume of the largest cylinder you can send by mail?



$V = \pi r^2 h$

$g(h,r) = \cancel{2}h + 4r = 104$

max $V(h,r)$ subject to $g(h,r) = 104$

$\nabla V = \langle \pi r^2, 2\pi r h \rangle$

$\nabla g = \langle 1\cancel{2}, 4 \rangle$

$\nabla V = \lambda \nabla g$

$g = 104$

$$\left.\begin{aligned} \pi r^2 &= \cancel{2}\lambda \\ 2\pi r h &= 4\lambda \end{aligned}\right\} \quad \frac{\pi r^2}{2\pi r h} = \frac{\cancel{2}\lambda}{4\lambda} = \frac{1}{\cancel{2}4}$$

$\cancel{2}h + 4r = 104$

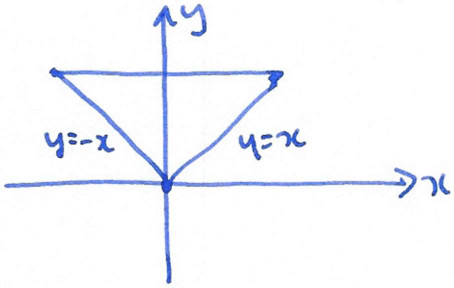
$\frac{r}{2h} = \frac{1}{4}$ $\quad 2r = h$

$6h = 104$

$h = \frac{104}{6}, \quad r = \frac{104}{\cancel{6}12}$

$V = \frac{\pi}{24}\left(\frac{104}{6}\right)^3.$

(3) (10 points) Find the integral of the function $f(x, y) = x - y$ over the triangle with vertices $(0, 0), (1, 1)$ and $(-1, 1)$.



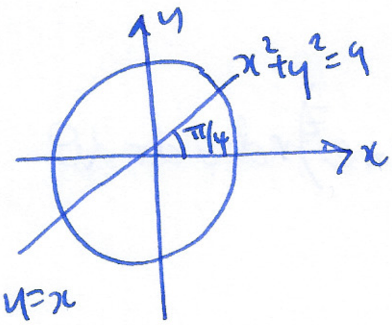
$$\int_0^1 \int_{-y}^{y} x-y \, dx \, dy$$

$$\left[\frac{1}{2}x^2 - xy\right]_{-y}^{y} = \frac{1}{2}y^2 - y^2 - \left(\frac{1}{2}y^2 + y^2\right)$$

$$\int_0^1 -2y^2 \, dy = \left[-\frac{2}{3}y^2\right]_0^1 = -\frac{2}{3}.$$

(4) (10 points) Write down limits for the integral of the function $f(x, y) = xy^2$ inside the circle $x^2 + y^2 = 9$, and above the line $y = x$.

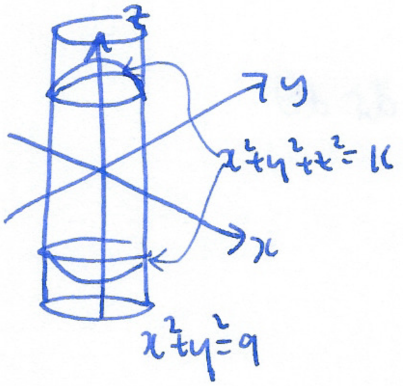
Do not evaluate this integral.



$$\int_{\pi/4}^{5\pi/4}\int_0^3 r\cos\theta \, r^2\sin^2\theta \; r\,dr\,d\theta$$

(5) (10 points) Write down limits for the integral of the function $f(x, y, z) = xy + e^z$ over the region inside both the cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 16$.

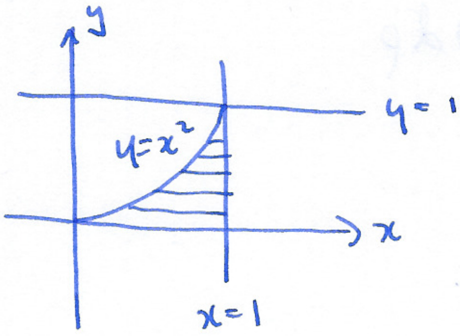
Do not evaluate the integral.



$$\int_0^{2\pi}\int_0^{3}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} \left(r^2\cos\theta\sin\theta + e^z\right) r\,dz\,dr\,d\theta$$

(6) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_{\sqrt{y}}^1 e^{-x^3}\, dx\, dy$$

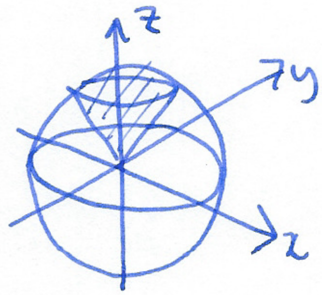


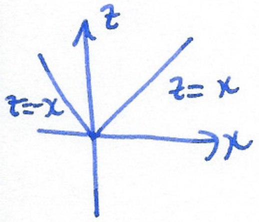
$$\int_0^1 \int_0^{x^2} e^{-x^3}\, dy\, dx$$

$$\left[y e^{-x^3} \right]_0^{x^2} = x^2 e^{-x^3}.$$

$$\int_0^1 x^2 e^{-x^3} dx = \left[-\frac{1}{3} e^{-x^3} \right]_0^1 = -\frac{1}{3} e^{-1} + \frac{1}{3}.$$

(7) (10 points) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 9$ above the positive cone $z = \sqrt{x^2 + y^2}$.





$$\int_0^3 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$\int_0^3 \rho^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi .$$

$$\left[\frac{1}{3}\rho^3\right]_0^3 \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/4} .$$

$$9 \cdot 2\pi \cdot \left(-\frac{1}{\sqrt{2}} + 1\right) .$$

(8) (10 points) Integrate the vector field $\mathbf{F} = \langle x^2, -z, y\rangle$ along the straight line from $\langle -2, -1, 1\rangle$ to $\langle 1, 5, 7\rangle$.

$$\underline{c}(t) = \langle -2, -1, 1\rangle + t\langle 3, 6, 6\rangle$$

$$\underline{c}'(t) = \langle 3, 6, 6\rangle$$

$$\int \underline{F}.\underline{ds} = \int_0^1 \langle (-2+3t)^2, -1-6t, -1+6t\rangle . \langle 3, 6, 6\rangle\, dt$$

$$= \int_0^1 27t^2 - 18t + 12 - 6 - 36t - 6 + 36t\, dt$$

$$= \int_0^1 27t^2 - 18t\, dt = \left[9t^3 - 9t^2\right]_0^1 = 0$$

$F_1 \quad F_2 \quad F_3$.

(9) (10 points) Show that the vector field $\mathbf{F} = \langle \sin(y), x\cos(y) + z, y \rangle$ is conservative, and find the potential function. Use this to evaluate

$$\int_C \mathbf{F}.\mathbf{ds},$$

where C is the curve from $(0, 2, 0)$ to $(3, 4, 12)$ formed by the intersection of the surfaces $z = xy$ and $z = 2x + 3y - 6$.

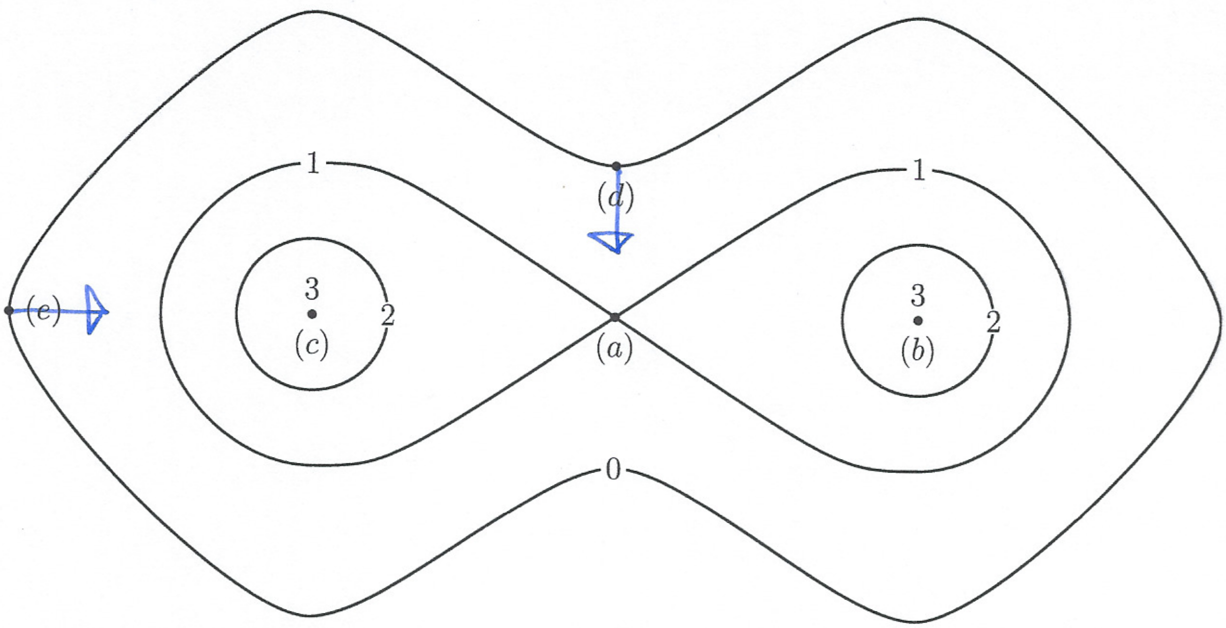
$\frac{\partial F_1}{\partial y} = \cos(y) \quad \frac{\partial F_2}{\partial x} = \cos(y) \checkmark \quad \frac{\partial F_1}{\partial z} = 0 \quad \frac{\partial F_3}{\partial x} = 0 \checkmark \quad \frac{\partial F_2}{\partial z} = 1 \quad \frac{\partial F_3}{\partial y} = 1 \checkmark$.

$\int F_1 \, d\cancel{y}x = x\sin(y) + c_1(y,z) \quad \int F_2 \, dy = x\sin(y) \overset{+zy}{} + c_2(x,z) \quad \int F_3 dz = zy + c_3(x,y)$

$f(x,y,z) = x\sin(y) + zy + c$

$\int_C \underline{F}.\underline{ds} = f(3,4,12) - f(0,2,0) = 3\sin(4) + 48 - 0$.

(10) (10 points) For each labelled point, either draw in the gradient vector, or, if it is a critical point, describe the type of critical point.



(a) saddle

(b) max

(c) max

(d)

(e)