# Sample midterm 2 Solutions



Q1 a) $f(x,y) = x^3 + 2y^3 - xy$

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 - y = 0\\ \frac{\partial f}{\partial y} &= 6y^2 - x = 0\end{aligned}\right\}\quad \begin{aligned} &6(3x^2)^2 = 0\\ &54x^4 - x = 0\end{aligned} \qquad \begin{aligned} &x(54x^3 - 1) = 0\\ &x = 0,\ \frac{1}{\sqrt[3]{54}} = \frac{1}{3\sqrt[3]{2}}\end{aligned}$$

$x = 0 \Rightarrow y = 0 : (0,0)$ $\quad x = \frac{1}{3\cdot 2^{1/3}} \Rightarrow y = \frac{1}{3\cdot 2^{2/3}}$ $\quad \left(\frac{1}{3}\cdot\frac{1}{2^{1/3}}, \frac{1}{3}\cdot\frac{1}{2^{2/3}}\right)$.

$f_{xx} = 6x$ $\quad D = f_{xx}f_{yy} - f_{xy}^2$ $\quad D(0,0) = 0 - (-1)^2 = -1 < 0 \Rightarrow$ saddle.

$f_{xy} = -1$

$f_{yy} = 6y$ $\quad D\left(\frac{1}{3\cdot 2^{1/3}}, \frac{1}{3\cdot 2^{2/3}}\right) = 6\cdot\frac{1}{3\cdot 2^{1/3}}\cdot 6\cdot\frac{1}{3\cdot 2^{2/3}} - (-1)^2$

$$= \frac{2}{2^{1/3}}\cdot\frac{2}{2^{2/3}} - 1 = 2 - 1 = 1 > 0$$

$f_{xx} > 0 \Rightarrow$ local min.

b) $f(x,y) = e^{x+y} - xe^{2y}$

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= e^{x+y} - e^{2y} = 0\\ \frac{\partial f}{\partial y} &= e^{x+y} - 2xe^{2y} = 0\end{aligned}\right\}\quad \begin{aligned} &e^y(e^x - e^y) = 0 &&\Rightarrow x = y\\ &e^y(e^x - 2xe^y) = 0 &&\Rightarrow e^{2x}(1-2x) = 0 \Rightarrow x = \tfrac{1}{2}.\end{aligned}$$

$\left(\frac{1}{2}, \frac{1}{2}\right)$.

$f_{xx} = e^{x+y}$ $\quad D = f_{xx}f_{yy} - f_{xy}^2$

$f_{xy} = e^{x+y} - 2e^{2y}$ $\quad D\left(\frac{1}{2}, \frac{1}{2}\right) = e\cdot(e - 2e) - (e - 2e)^2$

$f_{yy} = e^{x+y} - 4xe^{2y}$ $\quad = -e^2 - (-e)^2 = -2e^2 < 0$ saddle.

c) $f(x,y) = y\ln(x+y)$.

$$\left.\begin{aligned} f_x &= \frac{y}{x+y} = 0 &&\Rightarrow y = 0\\ f_y &= \ln(x+y) + \frac{y}{x+y} = 0 &&\Rightarrow x = 1\end{aligned}\right\}\qquad (1,0)$$

$f_{xx} = -y(x+y)^{-2}$ $\quad D = f_{xx}f_{yy} - f_{xy}^2$.

$f_{xy} = (x+y)^{-1} + y\cdot -(x+y)^{-2}$ $\quad D(1,0) = 0\cdot(*) - 1 = -1 < 0$ saddle.

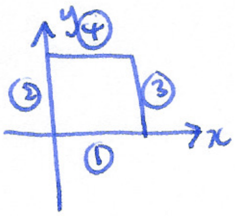
$f_{yy} = \frac{1}{x+y} + (x+y)^{-1} + y\cdot -(x+y)^{-2}$

**Q2** $f(x,y) = x^3 - xy - y^2 + y$

$f_x = 3x^2 - y = 0$, $f_y = -x - 2y + 1 = 0$ $\Big\}$ $y = 3x^2$, $-x - 6x^2 + 1 = 0$ $\quad (3x-1)(2x+1) = 0 \quad x = 1/3, -1/2$

$(1/3, 1/3), (-1/2, 3/4)$ $\quad f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27} - \frac{1}{9} - \frac{1}{9} + \frac{1}{3} = \frac{1-6+9}{27} = \frac{4}{27}$.

$f\left(-\frac{1}{2}, \frac{3}{4}\right) = -\frac{1}{8} - \frac{3}{8} - \frac{9}{16} + \frac{3}{4} = \frac{-2-6-9+12}{16} = -\frac{5}{16}$




check boundary: ② $x = 0 : f = -y^2 + y \quad f_y = -2y + 1 \quad y = \frac{1}{2}$.

$f\left(0, \frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.

① $y = 0 : f = x^3 \quad f_x = 3x^2$ $\qquad$ ③ $x = 1 \quad f(y) = 1 - y^2 \quad f_y = -2y \quad y = 0$.

④ $y = 1 : f(x) = x^3 - x \quad f_x = 3x^2 - 1 \quad x = \frac{1}{\sqrt{3}}$. $\quad f\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} - 1 + 1 = -\frac{2}{3\sqrt{3}}$.

corners: $f(0,0) = 0 \quad f(0,1) = 0 \quad f(1,0) = 1 \quad f(1,1) = 0$

biggest value: 1, smallest value $-5/16$.

**Q3**. min $f(x,y) = xy$ subject to $g(x,y) = 4 + 5x - y = 0$.

$\nabla f = \langle y, x \rangle \quad \nabla g = \langle 5, -1 \rangle$

$\nabla f = \lambda \nabla g$: $\quad y = 5\lambda$, $x = -\lambda$, $4 + 5x - y = 0$ $\Big\}$ $\quad 4 + 5(-\lambda) - (5\lambda) = 0$

$\lambda = \frac{4}{10} = \frac{2}{5} \qquad x = -\frac{2}{5}, \; y = 2$

$f\left(-\frac{2}{5}, 2\right) = -\frac{4}{5}$.

**Q4**


$V = \pi r^2 h$.
$A = \pi r^2 + 2\pi r h$

min $A(h,r) = \pi r^2 + 2\pi r h$ subject to $V = \pi r^2 h$

$\nabla A = \langle 2\pi r, 2\pi r + 2\pi h \rangle$

$\nabla V = \langle \pi r^2, 2\pi r h \rangle$

$\nabla A = \lambda \nabla V$:
$V = \pi r^2 h$

$2\pi r = \lambda \pi r^2$
$2\pi r + 2\pi h = \lambda 2\pi r h$
$\pi r^2 h = V$
$h = \frac{V}{\pi r^2}$

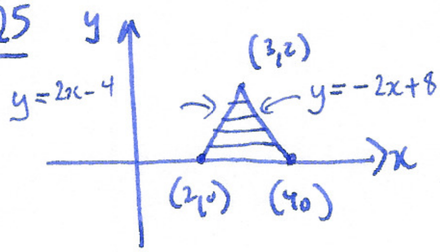
$\pi r (2 - \lambda r) = 0 \rightsquigarrow$ gives $r = 0$ or $r = 2/\lambda \Leftrightarrow \lambda = 2/r$

$2\pi r + 2\pi \cdot \frac{V}{\pi r^2} = \lambda 2\pi r \cdot \frac{V}{\pi r^2}$

$2\pi r + \frac{2V}{r^2} = \frac{4V}{r^2}$ $\qquad r^3 = \frac{2V}{2\pi} = \frac{V}{\pi}$ $\qquad r^3 = \sqrt[3]{V/\pi}.$

$h = \frac{V}{\pi r^2} = \frac{V}{\pi\sqrt[3]{V/\pi}}$ so $A = \pi\left(\frac{V}{\pi}\right)^{2/3} + 2\pi\sqrt[3]{V/\pi}\cdot\frac{V}{\pi\cdot\sqrt[3]{V/\pi}} = \pi^{1/3}V^{2/3} + 2V$

<u>Q5</u>



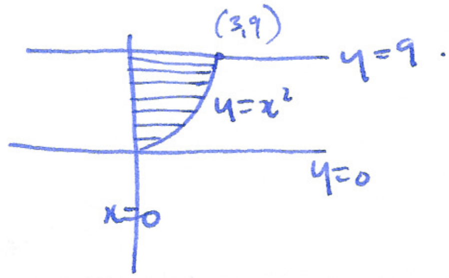

$$\int_0^2 \int_{\cancel{2x-4}\ \frac{y+4}{2}}^{\cancel{-2x}\ \frac{y-8}{-2}} xy\ dx\ dy \qquad \left[\frac{1}{2}x^2y\right]_{y/2+2}^{-y/2+4}$$

$$= \frac{1}{2}\left(-\frac{y}{2}+4\right)^2 y - \frac{1}{2}\left(\frac{y}{2}+2\right)^2 y = \frac{1}{2}y\left(\frac{y}{4} - 4y + 16 - \frac{y}{4} - 2y - 4\right)$$

$$= \frac{1}{2}y(-6y+12) = -3y^2+6y$$

$$\int_0^2 -3y^2+6y\ dy = \left[-y^3+3y^2\right]_0^2 = -8+12 = 4.$$

<u>Q6</u> $\displaystyle\int_0^9\int_0^{\sqrt{y}} \frac{x}{\sqrt{x^2+y}}\,dxdy$

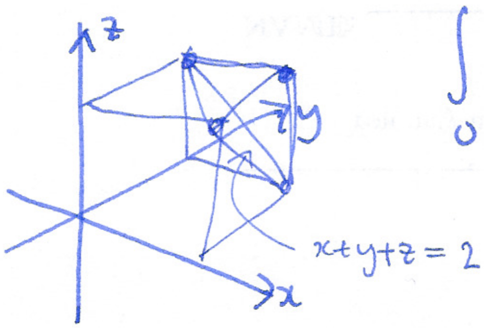



$$\int_0^3\int_{x^2}^9 \frac{x}{\sqrt{x^2+y}}\,dydx \qquad \left[2x.(x^2+y)^{1/2}\right]_{x^2}^9 = 2x(x^2+9)^{1/2} - 2x(2x^2)^{1/2}$$

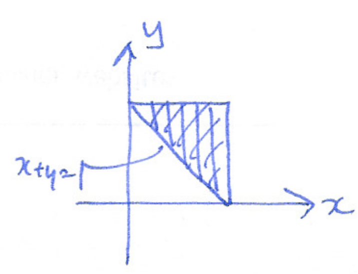
$$\int_0^3 2x\sqrt{x^2+9} - 2\sqrt{2}x^2\,dx = \left[\frac{1}{2}\frac{2}{3}(x^2+9)^{3/2} - 2\sqrt{2}\frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}18^{3/2} - 2\sqrt{2}.9 - \frac{2}{3}9^{3/2}.$$

<u>Q7</u>

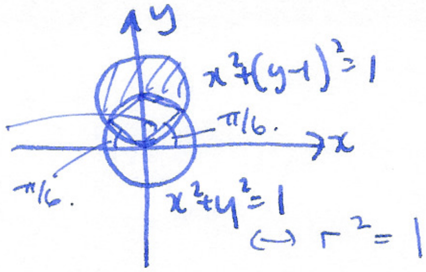



$$\int_0^1\int_{1-x}^1\int_{2-y-x}^1 dzdy\,dx.$$



**Q8** a)

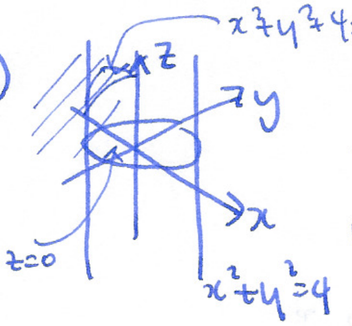
$\frac{\pi}{3}$

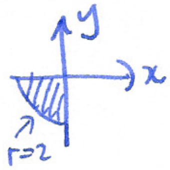



$$\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} r\,dr\,d\theta$$

$x^2+y^2-2y+1=1 \qquad r^2=2y=2r\sin\theta \qquad r=2\sin\theta$

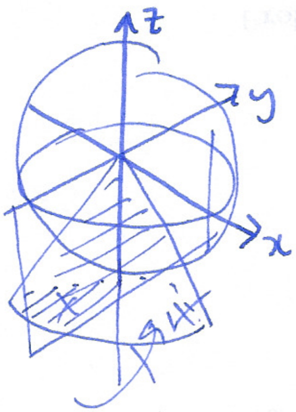
b)



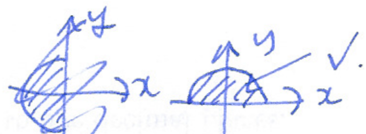



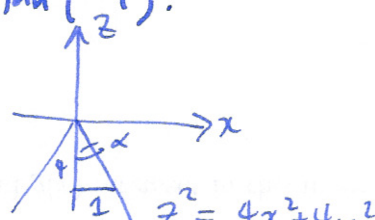

$$\int_{\pi}^{3\pi/2}\int_{0}^{2}\int_{0}^{\sqrt{4-r^2/4}} r\,dz\,dr\,d\theta.$$

c)




$$\int_{0}^{6}\int_{0}^{\pi}\int_{\pi-\tan^{-1}(1/4)}^{\pi} \rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$



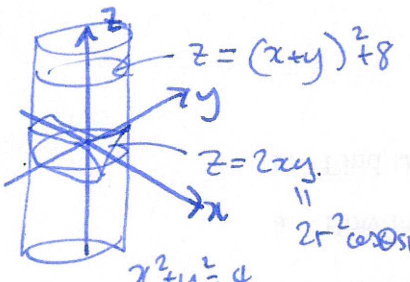




$z^2=4x^2+4y^2.$

$z^2=4y^2.$

$z=4y \qquad \alpha=\tan^{-1}(1/4).$

**Q9**




$\parallel$

$2r^2\cos\theta\sin\theta$

$$\int_{0}^{2}\int_{0}^{2\pi}\int_{\cancel{2xy}\;2r^2\sin\theta\cos\theta}^{r^2(\cos\theta+\sin\theta)^2+8} 1\;r\,dz\,d\theta\,dr$$

$$\Big[zr\Big]_{2r^2\sin\theta\cos\theta}^{r^2(\cos\theta+\sin\theta)^2+8} = r^3(\cos\theta+\sin\theta)^2+8-2r^3\sin\theta\cos\theta.$$

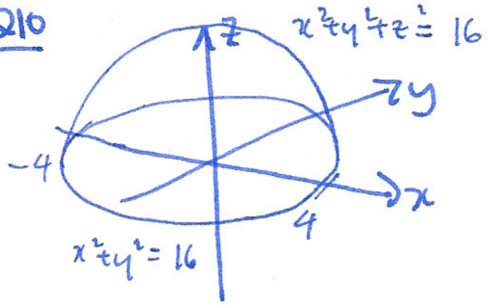
$$= r^3\left(\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta-2\sin\theta\cos\theta\right)$$

$$= r^3$$

$$\int_0^{2\pi} r^3\,d\theta = \left[\theta r^3\right]_0^{2\pi} = 2\pi r^3$$

$$\int_0^2 2\pi r^3\,dr = \left[\frac{\pi}{2} r^4\right]_0^2 = 8\pi$$

Q10




$$\int_0^4 \int_0^{2\pi} \int_0^{\pi/2} e^{-\rho^3} \rho^2 \sin\phi \; d\phi\, d\theta\, d\rho$$

$$\rho^2 e^{-\rho^3}\left[-\cos\phi\right]_0^{\pi/2} = \rho^2 e^{-\rho^3}(0-1) = -\rho^2 e^{-\rho^3}.$$

$$\left[-\rho^2 e^{-\rho^3}\theta\right]_0^{2\pi} = -2\pi\,\rho^2 e^{-\rho^3}.$$

$$\left[-2\pi\, e^{-\rho^3}\cdot -\frac{1}{3}\right]_0^4 = \frac{2}{3}\pi\left(-e^{-64}+1\right) = \frac{2}{3}\pi\left(1-e^{-64}\right).$$

Q11 $f(x,y,z) = e^z + xy$ $\quad \int_C f\,ds$ $\qquad C: \; c(t) = (-1,2,-2)+t(4,3,6)$

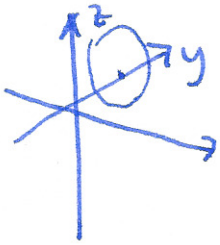
$$c'(t) = (4,3,6).$$

$$\int_C f\,ds = \int_0^1 f(c(t))\,\|c'(t)\|\,dt \qquad \|c'(t)\| = \sqrt{16+9+36} = \sqrt{61}.$$

$$= \int_0^1 \left(e^{-2+6t} + \underbrace{(-1+4t)(2+3t)}_{-2+5t+12t^2}\right)\sqrt{61}\,dt$$

$$= \sqrt{61}\left[e^{-2+6t}\cdot\frac{1}{6} - 2t + \frac{5}{2}t^2 + 4t^3\right]_0^1 = \sqrt{61}\left(\frac{1}{6}e^4 - 2 + \frac{5}{2} + 4 - \frac{1}{6}e^{-2}\right).$$

**Q12** $\underline{F} = \langle y, x^2, -z \rangle$ $\quad \dfrac{\partial F_1}{\partial y} = 1 \quad \dfrac{\partial F_2}{\partial x} = 2x \quad 1 \neq 2x \Rightarrow$ not conservative.




$C: \; c(\theta) = (2\cos\theta, 2, 2\sin\theta) \qquad 0 \leq \theta \leq 2\pi$

$c'(\theta) = \langle -2\sin\theta, 0, 2\cos\theta \rangle$

$$\int_C \underline{F}.\underline{ds} = \int_0^{2\pi} \langle 2, 4\cos^2\theta, -2\sin\theta \rangle . \langle -2\sin\theta, 0, 2\cos\theta \rangle \, d\theta$$

$$= \int_0^{2\pi} -4\sin\theta - \underbrace{4\sin\theta\cos\theta}_{2\sin 2\theta} \, d\theta = \Big[ +4\cos\theta + \cos 2\theta \Big]_0^{2\pi} = 0 .$$

**Q13** $\underline{F} = \langle y\cos(xy), ze^y + x\cos(xy), e^y \rangle$.

$\dfrac{\partial F_1}{\partial y} = \cos(xy) + -y.\sin(xy).x \qquad \dfrac{\partial F_2}{\partial x} = \cos(xy) + x.-\sin(xy).y \quad ✓$

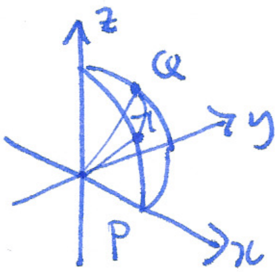
$\dfrac{\partial F_1}{\partial z} = 0 \qquad \dfrac{\partial F_3}{\partial x} = 0 \quad ✓ \qquad \dfrac{\partial F_2}{\partial z} = e^y \qquad \dfrac{\partial F_3}{\partial y} = e^y \quad ✓$

$\int y\cos(xy)\,dx = \sin(xy) + c_1(y,z)$

$\int ze^y + x\cos(xy)\,dy = ze^y + \sin(xy) + c_2(x,z)$

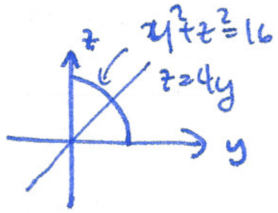
$\int e^y\,dz = ze^y + c_3(x,y)$

so $f(x,y,z) = ze^y + \sin(xy) + c$.



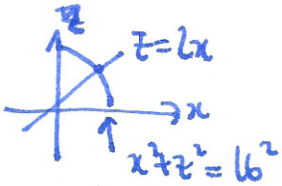

$\int_C \underline{F}.\underline{ds} = f(Q) - f(P)$

Q:




$\left( 0, \dfrac{16}{\sqrt{17}}, \dfrac{16.4}{\sqrt{17}} \right)$.

P:

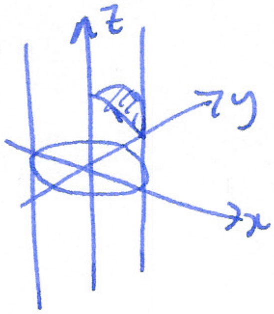



$\left( \dfrac{16}{\sqrt{5}}, 0, \dfrac{2.16}{\sqrt{5}} \right)$

$$f(Q)-f(P) = \frac{64}{\sqrt{17}} e^{16/\sqrt{17}} + \sin(0) - \frac{32}{\sqrt{5}} e^{0} - \sin(0)$$

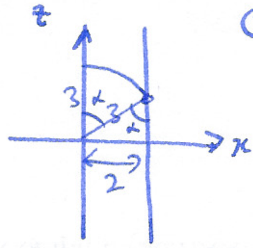
$$= \frac{64}{\sqrt{17}} e^{16/\sqrt{17}} - \frac{32}{\sqrt{5}} .$$

Q14




$$\Phi(\theta,\phi) = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi).$$

$$0 \le \theta \le \pi/2$$




$$0 \le \phi \le \sin^{-1}(2/3).$$

$$\Phi_\theta = (-3\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0)$$

$$\Phi_\theta \times \Phi_\phi = (-9\cos\theta\sin^2\phi, -9\sin\theta\sin^2\phi, -9\sin^2\theta\sin\phi\cos\phi - 9\cos^2\theta\sin\phi\cos\phi)$$

$$\Phi_\phi = (3\cos\theta\cos\phi, 3\sin\theta\cos\phi, -3\sin\phi) \qquad = 9(\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi).$$

$$\|\Phi_\theta \times \Phi_\phi\| = \sqrt{9(\sin^4\phi + \sin^2\phi\cos^2\phi)} = 3\sin\phi.$$

a) $$\iint_S (x^2+y^2)z\,dA = \int_0^{\sin^{-1}(2/3)}\int_0^{\pi/2} \underbrace{(9\cos^2\theta\sin^2\phi + 9\sin^2\theta\sin^2\phi)}_{9\sin^2\phi.} 3\cos\phi\, 3\sin\phi \, d\theta\, d\phi.$$

$$= \frac{\pi}{2}\int_0^{\sin^{-1}(2/3)} 81\cos\phi\sin^3\phi\, d\phi = \frac{81\pi}{8}\left[\sin^4\phi\right]_0^{\sin^{-1}(2/3)} = \frac{81\pi}{8}\cdot\left(\frac{2}{3}\right)^4.$$

b) $$\iint_S \langle x, 2y, 3z\rangle \cdot d\mathbf{S} = \int_0^{\sin^{-1}(2/3)}\int_0^{\pi/2} \langle 3\cos\theta\sin\phi, 6\sin\theta\sin\phi, 9\cos\phi\rangle \cdot 9\langle\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi\rangle \, d\theta\, d\phi$$

$$= 27\int_0^{\sin^{-1}(2/3)}\int_0^{\pi/2} \cos^2\theta\sin^3\phi + 2\sin^2\theta\sin^3\phi + 3\sin\phi\cos^3\phi \, d\theta\, d\phi. \ \ldots$$