Math 233 Calculus 3 Fall 18 Midterm 1a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find an expression for the angle between the two vectors $\langle 2, -3, 1\rangle$ and $\langle 1, 4, -2\rangle$.
(You can leave your answer in terms of trigonometric functions, you don't need to find the answer as a decimal.)

$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\theta \qquad \theta = \cos^{-1}\left(\frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \, \|\underline{v}\|}\right)$

$$\theta = \cos^{-1}\left(\frac{2-12-2}{\sqrt{4+9+1}\,\sqrt{1+16+4}}\right) = \cos^{-1}\left(\frac{-12}{\sqrt{14}\,\sqrt{21}}\right)$$

(2) (10 points) Find the equation of the plane through the point $(4, -1, 5)$ which is perpendicular to the line $\mathbf{r}(t) = \langle 1 - t, 2t + 1, 3t - 2\rangle$.

$\underline{n} = \langle -1, 2, 3 \rangle$

$(\underline{x} - \underline{p}) \cdot \underline{n} = 0 \qquad (\langle x, y, z \rangle - \langle 4, -1, 5 \rangle) \cdot \langle -1, 2, 3 \rangle = 0$

$-x + 2y + 3z = 9$

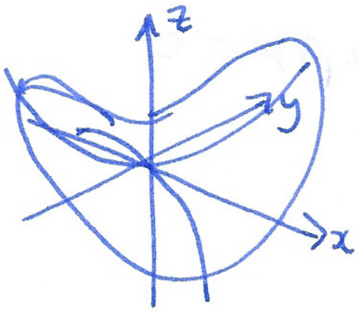
(3) (10 points) Sketch the graphs of the following functions, and their contour maps/level sets.

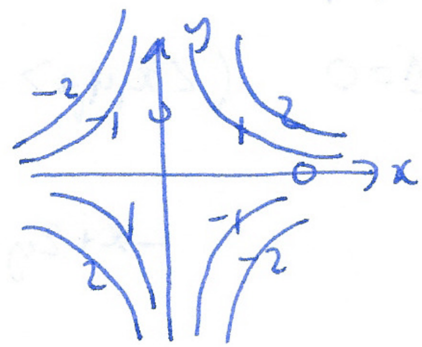
(a) $f(x, y) = xy$

(b) $f(x, y) = x^2 - 4y^2$

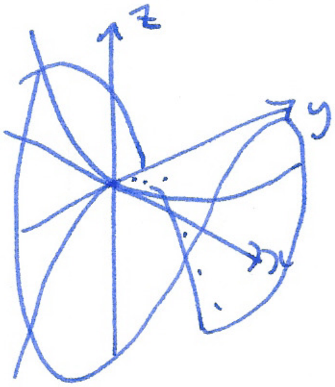
$xy = c$

a)

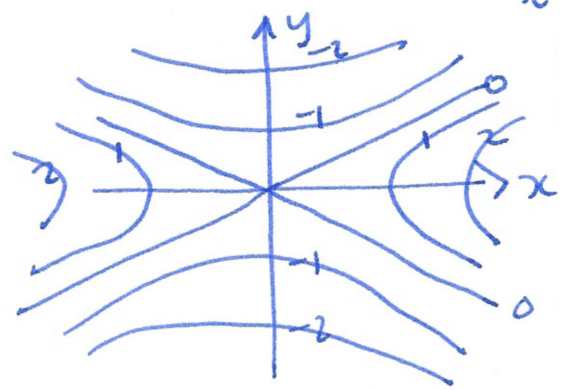




b)



$x^2 - 4y^2 = c$



(4) (10 points) Write down a parameterization for the straight line segment from $(1, -2, 2)$ to $(1, 3, -2)$. Use the integral formula for arc length to find the length of this line.

$\underline{r}(t) = \langle 1, -2, 2 \rangle + t\langle 0, 5, -4 \rangle \qquad 0 \leq t \leq 1$

$\underline{r}'(t) = \langle 0, 5, -4 \rangle$

$\|\underline{r}'(t)\| = \sqrt{41}$

arc length $= \int_0^1 \|\underline{r}'(t)\| \, dt = \int_0^1 \sqrt{41} \, dt = \left[\sqrt{41}\, t \right]_0^1 = \sqrt{41}$.

(5) (10 points) An object is thrown from the origin with initial velocity $\langle -10, 10, 20\rangle$ m/s. Find an expression for the position of the object at time t it moves under constant gravitational acceleration $\langle 0, 0, -g\rangle$ m/s^2. Feel free to take $g = 10$.

$$\underline{r}''(t) = \langle 0,0,-10\rangle$$

$$\underline{r}'(t) = \langle 0,0,-10\rangle t + \underline{v}_0 = \langle 0,0,-10\rangle t + \langle -10,10,20\rangle$$

$$\underline{r}(t) = \langle 0,0,-10\rangle \tfrac{1}{2}t^2 + \langle -10,10,20\rangle t + \underline{r}_0$$

$$= \langle 0,0,-5\rangle t^2 + \langle -10,10,20\rangle t$$

(6) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

x-axis : $\lim_{\substack{\cancel{x\to 0} \\ (x,0)\to(0,0)}} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$

y-axis : $\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$

$1 \neq -1 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

(7) Find the second order partial derivative f_{xz} for $f(x, y, z) = e^{3xyz} + \tan^{-1}(x - 2z)$.

$$f_x = e^{3xyz} \cdot 3yz + \frac{1}{1+(x-2z)^2} \cdot 1$$

$$f_{xz} = e^{3xyz} \cdot 3xy \cdot 3yz + e^{3xyz} \cdot 3y + -\left(1+(x-2z)^2\right)^{-2} \cdot 2(x-2z) \cdot (-2)$$

$f(2,1) = \ln(4+3) = \ln(7)$

(8) Find the linear approximation to $f(x,y) = \ln(2x + 3\sqrt{y})$ at the point $(2,1)$.

$f_x = \frac{1}{2x+3\sqrt{y}} \cdot 2$ $\quad$ $f_y = \frac{1}{2x+3\sqrt{y}} \cdot \frac{3}{2} y^{-1/2}$

$f_x(2,1) = \frac{2}{4+3} = \frac{2}{7}$ $\quad$ $f_y(2,1) = \frac{3}{2(4+3)} = \frac{3}{14}.$

$f(x,y) \approx \ln(7) + \frac{2}{7}(x-2) + \frac{3}{14}(y-1)$

(9) Find the normal vector to the surface $z^2 = x^2 + xy + 2y^2$ at the point $(2, 1, 8)$.

$F(x,y,z) = x^2 + xy + 2y^2 - z^2$

$\nabla F = \cancel{\langle -2z, x+4y,} \ \langle 2x+y,\ x+4y,\ -2z \rangle$

$\nabla F(2,1,8) = \langle 5, 6, -16 \rangle$

(10) The strength of a magnetic field is given by $B(x,y,z) = \dfrac{1}{(\sqrt{x^2+y^2+z^2})^3}$. $= (x^2+y^2+z^2)^{-3/2}$

Use the chain rule to find the rate of change of the field with respect to time at $t=2$ if you are travelling on the path $\mathbf{c}(t) = \langle 2t, t+1, 1-t \rangle$.

$$\nabla B = \left\langle -\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \cdot 2x,\ -\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \cdot 2y,\ -\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \cdot 2z \right\rangle$$

$$= \frac{-3 \langle x,y,z \rangle}{(x^2+y^2+z^2)^{5/2}}$$

$$\underline{c}'(t) = \langle 2,1,-1 \rangle$$

$$\frac{d}{dt}\left(B(\underline{c}(t))\right) = \nabla B(\underline{c}(t)) \cdot \underline{c}'(t) \qquad \underline{c}(2) = \langle 4,3,-1 \rangle$$

$$\frac{d}{dt}\left(B(\underline{c}(2))\right) = \frac{-3\langle 4,3,-1 \rangle}{26^{5/2}} \cdot \langle 2,1,-1 \rangle = \frac{-30}{(26)^{5/2}}$$