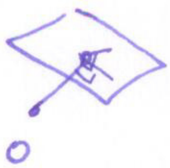
Solutions

Q1 a) $\underline{n} = \langle -1,1,-1\rangle \times \langle 1,0,0\rangle = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0,-1,-1\rangle$

b) $(\underline{x}-\underline{a})\cdot\underline{n} = 0$ $\qquad (x-1)\cdot 0 + (y-2)\cdot(-1) + (z-2)(-1) = 0$

$y + z = 4$

Q2 a)



want $\underline{x}$ parallel to $\underline{n}$, so $(\lambda\underline{n} - \underline{a})\cdot\underline{n} = 0$

$$\lambda = \frac{\underline{a}\cdot\underline{n}}{\underline{n}\cdot\underline{n}} = \frac{\langle 1,2,2\rangle\cdot\langle 0,-1,-1\rangle}{2} = -2$$

$\underline{x} = \langle 0,2,2\rangle$. distance is $\sqrt{0^2+2^2+2^2} = 2\sqrt{2}$.

b) min $\underset{f(x,y,z)}{x^2+y^2+z^2}$ subject to $\underset{g(y,z)=4.}{y+z=4.}$

solve $\nabla f = \lambda\nabla g$ $\qquad \nabla f = \langle 2x, 2y, 2z\rangle \qquad \nabla g = \langle 0,1,1\rangle.$

$g = 4$

$2x = \lambda\cdot 0 \qquad x = 0$

$2y = \lambda\cdot 1$

$2z = \lambda\cdot 1 \qquad y = z$

$y+z=4.$ $\qquad 2y = 4 \quad y = 2,\ z = 2. \qquad \langle 0,2,2\rangle$, distance $2\sqrt{2}$.

Q3 $\underline{a} = \langle 1,-2,4\rangle \quad \underline{b} = \langle 1,2,-1\rangle$

$$\underline{x} = \frac{\underline{a}\cdot\underline{b}}{\underline{b}\cdot\underline{b}}\underline{b} = \frac{1-4-4}{1+4+1}\langle 1,+2,\overset{-1}{\cancel{4}}\rangle = \frac{-7}{6}\langle 1,+2,\overset{-1}{\cancel{4}}\rangle.$$

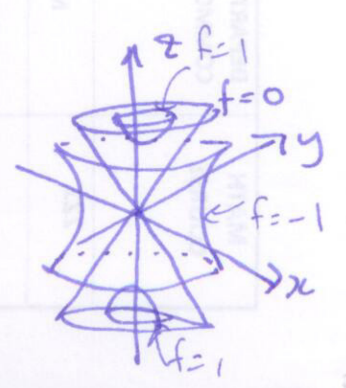
$$\underline{y} = \langle 1,-2,4\rangle + \frac{7}{6}\langle 1,+2,\overset{-1}{\cancel{4}}\rangle.$$

Q4 $f(x,y,z) = x^2+y^2-z^2$

$\nabla f = \langle 2x, 2y, -2z\rangle$

$\nabla f(2,2,1) = \langle 4,4,-2\rangle$ $\qquad 4x+4-z = 14.$

tangent plane $(x-2)\cdot 4 + (y-2)\cdot 4 + (z-\overset{1}{\cancel{4}})\cdot(-2) = 0$



Q5: $\underline{c}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle$

$$\underline{c}'(t) = \langle 2t, \cos t - \cos t - t.-\sin t, -\sin t + \sin t + t\cos t \rangle$$
$$= \langle 2t, t\sin t, t\cos t \rangle$$

$$\|\underline{c}'(t)\| = t\sqrt{4 + \sin^2 t + \cos^2 t} = \sqrt{5}\,t.$$

$$\text{arc length} = \int_0^\pi \|\underline{c}'(t)\| dt = \int_0^\pi \sqrt{5}\,t\, dt = \left[\frac{\sqrt{5}}{2} t^2\right]_0^\pi = \frac{\sqrt{5}}{2}\pi^2.$$

Q6 $\underline{v} = \langle -t, 3t^2, 6t^3 \rangle$

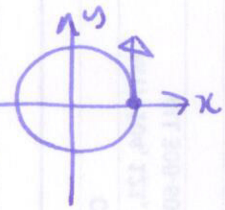
$$\underline{x} = \langle -\tfrac{1}{2}t^2, t^3, \tfrac{3}{2}t^4 \rangle + \underline{c}$$

$$\underline{x}(0) = \underline{0} + \underline{c} = \langle 2, -3, 7 \rangle$$

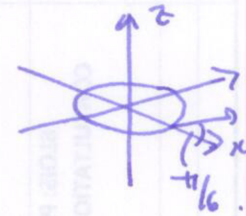
$$\underline{x}(t) = \langle 2, -3, 7 \rangle + \langle -\tfrac{1}{2}t^2, t^3, \tfrac{3}{2}, t^4 \rangle$$

$$\underline{x}(3) = \langle 2, -3, 7 \rangle + \langle -\tfrac{1}{2}.9, 27, \tfrac{3}{2}.81 \rangle = \langle -\tfrac{5}{2}, 24, \tfrac{257}{2} \rangle$$

Q7



$$\underline{c}(t) = \langle 20\cos(\lambda t), 20\sin(\lambda t), 0 \rangle.$$
$$\underline{c}'(t) = \langle -20\lambda\sin(\lambda t), 20\lambda\cos(\lambda t), 0 \rangle.$$
$$\|\underline{c}'(t)\| = 20\lambda = 2 = \lambda = 1/10.$$
$$\underline{c}'(0) = \langle -20.\tfrac{1}{10}.0, \ 2\underbrace{\cos(0)}_{1}, 0 \rangle = \langle 0, 2, 0 \rangle.$$



$$\underline{x_0} = \underline{c}(0) = \langle 20, 0, 0 \rangle$$
$$\underline{v_0} = \langle 0, 2, 0 \rangle + 5\langle \cos(\pi/6), 0, \sin(\pi/6) \rangle = \langle \tfrac{5\sqrt{3}}{2}, 2, \tfrac{5}{2} \rangle$$

$$\underline{a} = \underline{x}''(t) = \langle 0, 0, -10 \rangle$$
$$\underline{v} = \underline{x}'(t) = \langle 0, 0, -10t \rangle + \underline{c} = \langle 0, 0, -10t \rangle + \langle \tfrac{5\sqrt{3}}{2}, 2, \tfrac{5}{2} \rangle.$$
$$\underline{x}(t) = \langle 0, 0, -5t^2 \rangle + \langle \tfrac{5\sqrt{3}}{2}, 2, \tfrac{5}{2} \rangle t + \underline{c}$$
$$= \langle 0, 0, -5t^2 \rangle + \langle \tfrac{5\sqrt{3}}{2}, 2, \tfrac{5}{2} \rangle t + \langle 20, 0, 0 \rangle$$

lands at $z = 0$: $-5t^2 + \frac{5}{2}t = 0 \quad t(-5t + \frac{5}{2}) = 0 \quad t = \frac{1}{2}$

lands at $\underline{x}(\tfrac{1}{2}) = \langle 0, 0, -5/4 \rangle + \langle \tfrac{5\sqrt{3}}{4}, 1, \tfrac{5}{4} \rangle + \langle 20, 0, 0 \rangle = \langle 20 + \tfrac{5\sqrt{3}}{4}, 1, 0 \rangle$

Q8 $f(x,y,z) = e^{x-2y} + \tan(2yz)$

a) $\nabla f = < e^{x-2y}, -2e^{x-2y} + \sec^2(2yz)\cdot 2z, \sec^2(2yz)\cdot 2y >$

$\nabla f(2,0,1) = < e^2, -2e^2+2, 0 >$

b) tangent plane: $e^2(x-2) + (2-2e^2)(y-0) + 0\cdot(z-1) = 0$

Q9 $f(x,y) = 2x^2 - 2x + y^2 + 2$

a) $\left.\begin{array}{l} \frac{\partial f}{\partial x} = 4x-2 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0 \end{array}\right\}$ $\begin{array}{l} x=2 \\ y=0 \end{array}$ critical point $(2,0)$.

$f_{xx} = 4$
$f_{xy} = 0$
$f_{yy} = 2$

$D(2,0) = 4.2.-0 = 8 > 0$

$f_{xx} > 0 \Rightarrow$ local min.

b) $\nabla f = < 4x-2, 2y >$ $g(x,y) = x^2+y^2 = 9$

$\nabla g = < 2x, 2y >$

solve $\nabla f = \lambda \nabla g$
$g = 9$

$4x-2 = 2x\lambda \Rightarrow x = 1$
$2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$
$x^2+y^2 = 9$

$y = \pm 2\sqrt{2}$ $(1, \pm 2\sqrt{2})$. and $(\pm 3, 0)$.

c) $f(2,0) = 8-4+0+2 = 4 \leftarrow$ min

$f(1, 2\sqrt{2}) = 2-2+8+2 = 10$
$f(1, -2\sqrt{2}) = 2-2+8+2 = 10$
$f(3,0) = 18-6+0+2 = 12$
$f(-3,0) = 18+6+0+2 = 26 \leftarrow$ max.

Q10 $f(x,y) = 3xy - x^2y - xy^2$

$f_x = 3y - 2xy - y^2 = 0$ $y(3-2x-y) = 0$

$f_y = 3x - x^2 - 2xy = 0$ $x(3-x-2y) = 0$

④

$x=0: \quad y(3-y)=0 \qquad (0,0),(0,3)$

$y=0: \quad x(3-x)=0 \qquad (0,0)\ (3,0)$

$\left.\begin{array}{l} 3-2x-y=0 \ ① \\ 3-x-2y=0 \ ② \end{array}\right\}$ $②-2①: \ -3=-3x=0 \quad x=1, y=1 \quad (1,1)$

$D = f_{xx} f_{yy} - f_{xy}^2$

$f_{xx} = -2y$

$f_{xy} = 3-2x-2y$

$f_{yy} = -2x$

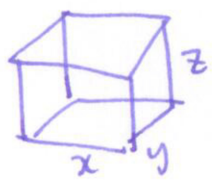
$D(0,0) = 0$ no information

$D(0,3) = 0-(-3)^2 < 0$ saddle

$D(3,0) = 0-(-3)^2 < 0$ saddle

$D(1,1) = 4-(-1)^2 > 0 \quad f_{xx} < 0 \Rightarrow$ local max.

Q11



$V = xyz = 6000 \qquad \nabla V = \langle yz, xz, xy \rangle$

$H = \cancel{4xy}\ 2xy + 6yz + 6xz \qquad \nabla H = \langle 2y+6z, 2x+6z, 6y+6x \rangle$

solve $\nabla H = \lambda \nabla V$
$V = 6000$

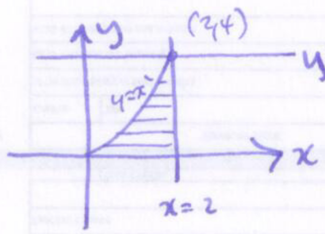
$\left.\begin{array}{l} 2y+6z = \lambda yz \\ 2x+6z = \lambda xz \\ 6y+6x = \lambda xy \\ xyz = 6000 \end{array}\right\}$ $2xy+6xz = 2xy+6yz = 6yz+6xz$

$6xz = 6yz \qquad 2xy = 6xz \qquad 2xy = 6yz.$

$x = y \qquad y = 3z \qquad x = 3z$

$x \cdot x \cdot \frac{x}{3} = 6000 \quad x = \sqrt[3]{18000} = y \quad z = \frac{1}{3}\sqrt[3]{18000}.$

Q12 $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 y \sin(x^5)\, dx\, dy$

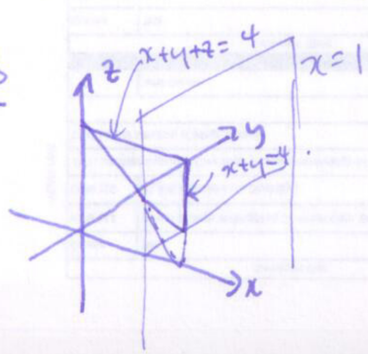


$\displaystyle\int_0^2 \int_0^{x^2} y \sin(x^5)\, dy\, dx$

$\left[\frac{1}{2} y^2 \sin(x^5)\right]_0^{x^2} = \frac{1}{2} x^4 \sin(x^5) \qquad \displaystyle\int_0^2 \frac{1}{2} x^4 \sin(x^5)\, dx$

$= \left[-\frac{1}{2}\cos(x^5) \cdot \frac{1}{5}\right]_0^2 = -\frac{1}{10}\cos(32) + \frac{1}{10}.$

Q13



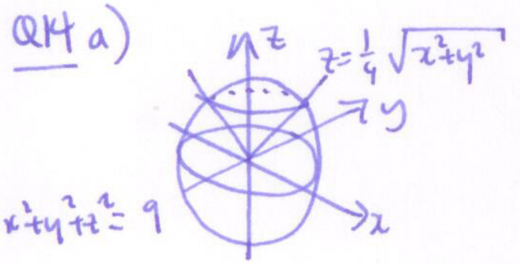
$\displaystyle\int_0^1 \int_0^{4-x} \int_0^{4-x-y} dz\, dy\, dx$

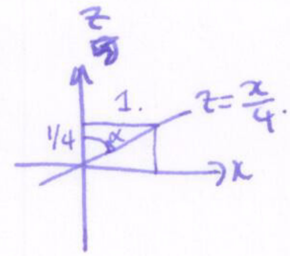
$\left[z\right]_0^{4-x-y} = 4-x-y \qquad \int_0^{4-x} 4-x-y\,dy = \left[4y-xy-\frac{1}{2}y^2\right]_0^{4-x}$

$\int_0^1 4(4-x)-x(4-x)-\frac{1}{2}(4-x)^2\,dx = \int_0^1 16-4x-4x+x^2-8+4x-\frac{1}{2}x^2\,dx$

$=\int_0^1 8-4x+\frac{1}{2}x^2\,dx = \left[8x-2x^2+\frac{1}{6}x^3\right]_0^1 = 8-2+\frac{1}{6} = \frac{37}{6}.$

Q14 a)

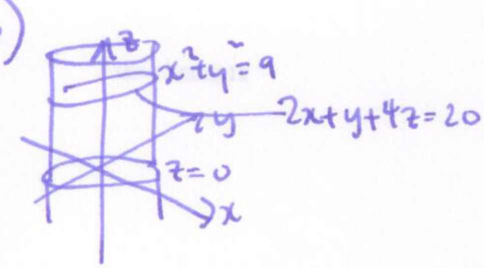




$\tan(\alpha)=4$

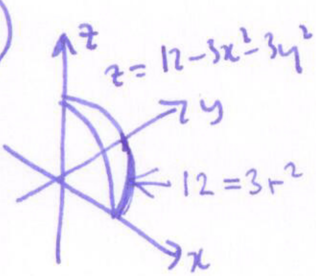
$$\int_0^{2\pi}\int_0^{3}\int_{\alpha=\tan^{-1}(4)}^{\pi} \rho^2\sin\phi\;d\phi\,d\rho\,d\theta$$

b)



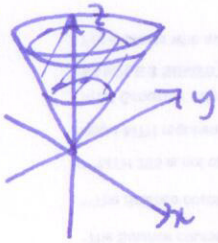
$$\int_0^{3}\int_0^{2\pi}\int_0^{\frac{20-2x-y}{4}} r\,dz\,d\theta\,dr$$

c)



$$\int_0^{2}\int_0^{\pi/2}\int_0^{12-3r^2} r\,dz\,d\theta\,dr$$

Q15

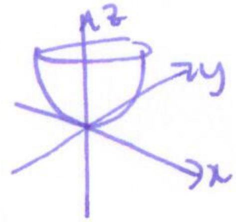


$$\int_1^{3}\int_0^{2\pi}\int_0^{\pi/3} \rho\cos\theta\sin\phi\;\rho\sin\theta\sin\phi\;\rho\cos\phi\;\rho^2\sin\phi\;d\phi\,d\theta\,d\rho.$$

$$= \int_1^3 \rho^5\,d\rho \int_0^{2\pi}\cos\theta\sin\theta\,d\theta \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi.$$

$\left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0 \qquad = 0$

Q15

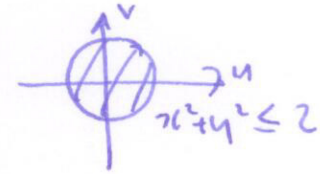


$\phi(u,v) = \langle u, v, u^2+v^2 \rangle \qquad \underline{F} = \langle y, -x, z^2 \rangle.$

$\phi_u = \langle 1, 0, 2u \rangle$

$\phi_v = \langle 0, 1, 2v \rangle$

$\underline{n} = \langle -2u, -2v, 1 \rangle$

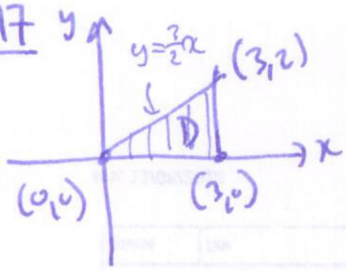


$$\iint_D \langle v, -u, (u^2+v^2)^2 \rangle . \langle -2u, -2v, 1 \rangle \, du\,dv = \iint_D -2uv + 2uv + (u^2+v^2)^2 \, du\,dv$$

change to polars in (u,v) plane: $\displaystyle\int_0^{2\pi}\int_0^2 r^4 \, r \, dr\,d\theta = \left[\frac{1}{6}r^6\right]_0^2 = \frac{32}{3}$

$$\int_0^{2\pi} \frac{32}{3} \, d\theta = \frac{64}{3}\pi.$$

Q17



$\underline{F} = \langle \tan(1+x^3), 2xy \rangle$

$$\int_C \underline{F}.\underline{ds} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx\,dy$$

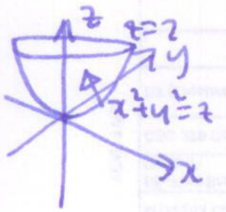
$$\int_0^3 \int_0^{\frac{3}{2}x} 2y - 0 \, dy\,dx = \left[y^2\right]_0^{\frac{3}{2}x} = \frac{9}{4}x^2.$$

$$\int_0^3 \frac{9}{4}x^2 \, dx = \left[\frac{3}{4}x^3\right]_0^3 = \frac{81}{4}.$$

Q18

$\underline{F} = \langle y^2, x, z^2 \rangle \qquad \int_{\underline{c}} \underline{F}.\underline{ds} \qquad \underline{c}(\theta) = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, 2 \rangle \quad 0 \le \theta \le 2\pi$

$\underline{c}'(\theta) = \langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0 \rangle.$



$$\int_0^{2\pi} \langle 2\sin^2\theta, \sqrt{2}\cos\theta, 4 \rangle . \langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0 \rangle \, d\theta.$$

$$\int_0^{2\pi} -2\sqrt{2}\underbrace{\sin^3\theta}_{\sin\theta(1-\cos^2\theta)} + \underbrace{2\cos^2\theta}_{1+\cos 2\theta} \, d\theta$$

$$\int_0^{2\pi} -2\sqrt{2}\sin\theta + 2\sqrt{2}\sin\theta\cos^2\theta + 1 + \cos 2\theta \, d\theta = \left[2\sqrt{2}\cos\theta - \frac{2\sqrt{2}}{3}\cos^3\theta + \theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 2\pi.$$

$\phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2) \quad 0 \le r \le \sqrt{2} \quad 0 \le \theta \le 2\pi$

$\phi_r = (\cos\theta, \sin\theta, 2r)$

$\phi_\theta = (-r\sin\theta, r\cos\theta, 0) \qquad \underline{n} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

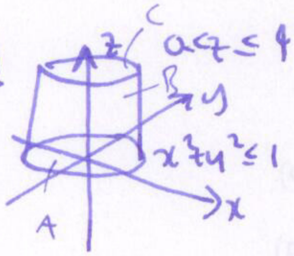
$\iint_S \underline{\nabla \times F}\, d\underline{S}$ $\qquad \underline{\nabla \times F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1-2y \rangle$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \langle 0, 0, 1 - 2r\sin\theta \rangle \cdot \langle \cancel{-2r^2\cos\theta}, -2r^2\sin\theta, r \rangle \, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} r - 2r^2\sin\theta \, dr\, d\theta \qquad \left[\tfrac{1}{2}r^2 - \tfrac{2}{3}r^3\sin\theta\right]_0^{\sqrt{2}} = \cancel{2}\,1 - \frac{\cancel{16}\,8}{3}\sin\theta$$

$$\int_0^{2\pi} \cancel{2}\,1 - \frac{\cancel{16}\,8}{3}\sin\theta \, d\theta = \left[\cancel{2}\theta + \frac{\cancel{16}\,8}{3}\cos\theta\right]_0^{2\pi} = \cancel{4\pi}.\ 2\pi.$$

Q19



$\underline{F} = \langle x, -y, z \rangle$ $\qquad \iint_{\partial W} \underline{F}\cdot\underline{dS} = \iiint_W \underline{\nabla}\cdot\underline{F}\, dV.$

$\underline{\nabla}\cdot\underline{F} = \cancel{\langle 1, -1, 1 \rangle}\ 1 - 1 + 1 = 1$

$\iiint_W 1\, dV = \text{vol}(W) = 4\pi.$

$\iint_{\partial W} \underline{F}\cdot\underline{dS}$: $\quad$ A: $\phi(r,\theta) = (r\cos\theta, r\sin\theta, 0) \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$

$\phi_r = (\cos\theta, \sin\theta, 0)$

$\phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\underline{n} = \langle 0, 0, r \rangle \leftarrow$ wrong direction!

$\iint_A \underline{F}\cdot\underline{dS} = \iint_A \langle r\cos\theta, -r\sin\theta, 0 \rangle \cdot \langle 0, 0, r \rangle \, dA = \iint 0 \, dr\, d\theta = 0.$

B: $\phi(\theta, z) = (\cos\theta, \sin\theta, z) \quad 0 \le \theta \le 2\pi \quad 0 \le z \le 4$

$\phi_\theta = (-\sin\theta, \cos\theta, 0)$

$\phi_z = (0, 0, 1)$

$\underline{n} = (\cos\theta, \cancel{\theta}\, +\sin\theta, 0)$

$$\int_0^4 \int_0^{2\pi} \langle \cos\theta, -\sin\theta, z \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle \, d\theta\, dz$$

$\int_0^4 \int_0^{2\pi} \cos^2\theta - \sin^2\theta \; d\theta \, dz$ $\qquad \int_0^{2\pi} \cos 2\theta \, d\theta = 0 \quad \int_0^4 0 \, dz = 0.$

C: $\phi(r,\theta) = (r\cos\theta, r\sin\theta, 4) \qquad 0 \le \theta \le 2\pi \qquad 0 \le r \le 1$

$\phi_r = (\cancel{\sin\theta}\ \cos\theta, \ \sin\theta, 0)$

$\phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\underline{n} = (0, 0, r)$ ← right direction.

$\int_0^1 \int_0^{2\pi} \langle r\cos\theta, -r\sin\theta, 4\rangle \cdot \langle 0, 0, r\rangle \, d\theta \, dr = \int_0^1 \int_0^{2\pi} 4r \, d\theta \, dr$

$= [4r\theta]_0^{2\pi} = 8\pi r \qquad \int_0^1 8\pi r \, dr = [4\pi r^2]_0^1 = 4\pi.$