§5.4 Fundamental theorem of calculus II

<u>Thm</u> (FTC 2) Let $f(x)$ be a continuous function on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is an anti-derivative for $f(x)$, i.e. $A'(x) = f(x) = \frac{dA}{dx}$

i.e. $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$. Furthermore $A(a) = 0$.

§5.6 Substitution | change of variable

"reverse chain rule for integration"

<u>recall</u> : chain rule : $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)$.

<u>substitution | change of variable</u>

$\int f(x)\,dx \quad \cancel{u(x)}\ \ x(u)$

$$\boxed{\int_{x=a}^{x=b} f(x)\,dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du}$$

<u>remember</u> : three things to change : limits, functions, differential.

<u>mnemonic</u> : $du = \frac{du}{dx} \cdot dx$ (cancelling fractions).

why does this work?

$\int f(x)\,dx$, $x(u)$ set $F(x) = \int f(x)\,dx$, so $F'(x) = f(x)$

$\int f(x)\,dx = F(x) \xrightarrow[\text{wrt } u]{\text{diff}} \frac{d}{du}\left(F(x(u)) \right) = F'(x(u)) \cdot x'(u)$.

$\xrightarrow[\text{wrt } u]{\text{integrate}} \quad \int f(x(u))\,x'(u)\,d\cancel{x}u = \int f(x(u))\,\frac{dx}{du}\,du \quad \square$.

useful fact : $\boxed{\frac{dx}{du} = 1 \Big/ \frac{du}{dx}}$

<u>Examples</u> $\int_{x=0}^{x=1} e^{-7x}\,dx$ set $u = -7x$

$\frac{du}{dx} = -7$

$\int_{u=0}^{u=-7} e^u \frac{dx}{du}\,du = \int_0^{-7} e^u \cdot -\frac{1}{7}\,du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}\left(e^{-7} - 1\right)$.

Examples $\int_0^2 x^2\sqrt{x^3+1}\,dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$, $\int_0^1 \frac{x}{x^2+1}\,dx$

§5.8 More integrals

recall: $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$ so $\int \frac{1}{x}\,dx = \ln|x| + c$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)+c$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x)+c$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \int e^x\,dx = e^x + c$$

recall: $b^x = e^{x\ln(b)}$

$$\int b^x\,dx = \int e^{x\ln(b)}\,dx = \frac{1}{\ln(b)}e^{x\ln(b)} + c$$

(sub $u = x\ln(b)$).

Example: $\int \frac{1}{9+x^2}\,dx = \frac{1}{9}\int \frac{1}{1+x^2/9}\,dx = \frac{1}{9}\int \frac{1}{1+(x/3)^2}\,dx$

substitute $u = \frac{x}{3}$ ...

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx \quad \text{sub } u = 2x \ldots$$

Examples $\int x(x+1)^9\,dx$ $\quad \int \sqrt{4x-1}\,dx$ $\quad \int \sin^2 x \cos x\,dx$

$\int x\cos(x^2)\,dx$ $\quad \int x\sqrt{x^2-1}\,dx$ $\quad \int \frac{1}{x\ln(x)}\,dx$ $\quad \int \frac{e^{2x}+e^{4x}}{e^x}\,dx$ $\quad \int 2^x\,dx$ $\quad \int xe^{x^2}\,dx$