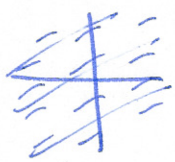
Example $\int x^2 + \frac{1}{x} + \sin(x)\,dx = \frac{1}{3}x^3 + \ln|x| - \cos(x) + c.$

Alternate view

we can think of finding the indefinite integral as finding a function given its slope function, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (sometimes called an initial condition) then this gives a particular solution.



$\frac{dy}{dx} = 1$

Example an object falls freely under gravity, so

acceleration: $a(t) = -g$ (constant)

velocity: $v(t)$ has $v'(t) = a(t)$

$$\frac{dv}{dt} = -g$$

has general solution $v(t) = -gt + c$

if the velocity at time $t=0$ is v_0, then $v(0) = v_0 = c$ and so the particular solution is $v(t) = -gt + v_0$

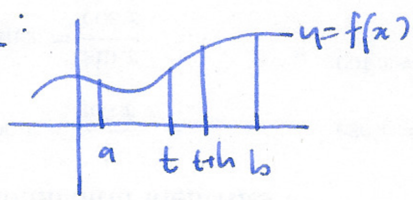
position: $x(t)$ has $\frac{dx}{dt} = v(t) = -gt + v_0$, has general solution

$x(t) = -\frac{1}{2}gt^2 + v_0t + c$. If position at time $t=0$ is x_0, then $x(t) = -\frac{1}{2}gt^2 + v_0t + x_0$

§5.4 Fundamental theorem of calculus

Thm (FTC ①) suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

intuition:



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0} \frac{\int_t^{t+h} f(x)\,dx}{h}$$

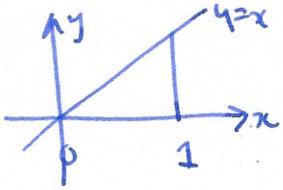
$$\underset{\text{approx}}{\approx} \frac{\text{area of rectangle } f(t)h}{h} = f(t)$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x)\,dx = F(t)+c$

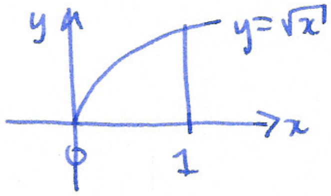
what is the constant? $t=a$: $\int_a^a f(x)\,dx = 0 = F(a)+c \Rightarrow c = -F(a)$

so $\int_a^t f(x)\,dx = F(t) - F(a)$ $\square$.

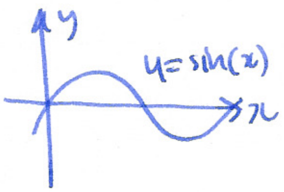
Examples



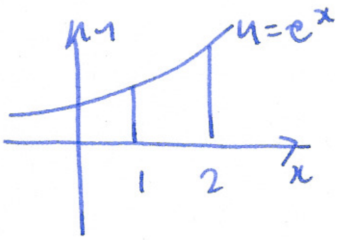
$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$



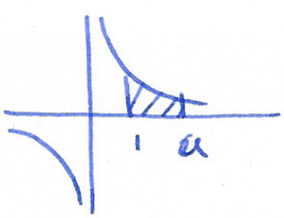
$$\int_0^1 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3} - 0 = \tfrac{2}{3}.$$



$$\int_0^\pi \sin(x)\,dx = \left[-\cos(x)\right]_0^\pi = -\cos(\pi) + \cos(0) = -(-1)+1 = 2$$



$$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2 - e^1$$



$$\int_1^a \frac{1}{x}\,dx = \left[\ln|x|\right]_1^a = \ln|a| - \ln|1| = \ln|a|.$$

Observations

① choice of antiderivative doesn't matter, let $F(x)$ and $F(x)+c$ be antiderivatives for $f(x)$. Then
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$= F(b)+c - (F(a)+c) = F(b) - F(a).$$

② $\int_a^t f(x)\,dx$ is a function of t! x is called a dummy variable.

i.e. $\int_a^t f(x)\,dx = \int_a^t f(y)\,dy$, if you want a function of x use $\int_a^x f(t)\,dt$.