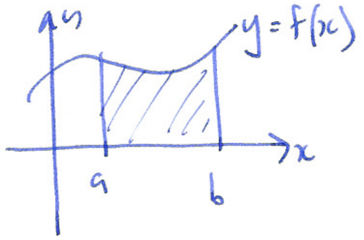
useful fact:

Thm If $f(x)$ is cts on $[a,b]$ then all of these approximations give the same limit as $N \to \infty$, which is the area under the curve.

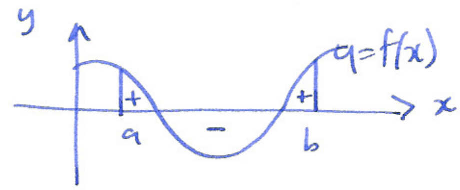
$$\text{area} = \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N$$

§5.2 Definite integral

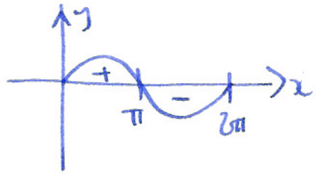


$\int_a^b f(x)\,dx =$ area under the curve $y=f(x)$ between $x=a$ and $x=b$

note: signed area

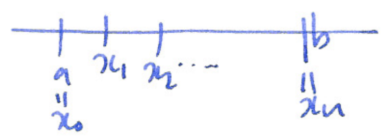


so $\int_0^{2\pi} \sin(x)\,dx = 0$



Formal definition: Riemann sum $R(f,P,c)$

$P =$ partition of $[a,b]$



widths $\Delta x_i = x_i - x_{i-1}$

$c =$ choice of points $c_i \in [x_{i-1}, x_i]$

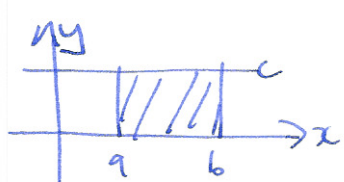
$$R(f,P,c) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i \qquad \|P\| = \max \Delta x_i$$

Defn $$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f,P,c) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

when this limit exists, we say f is integrable over $[a,b]$

Useful properties

$$\int_a^b c\,dx = c(b-a)$$



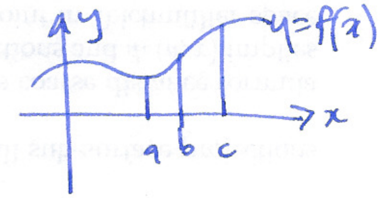
$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

0-length interval: $\int_a^a f(x)\,dx = 0$

adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



comparisons: if $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

§5.3 Antiderivatives

Defn A function $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Example if $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative.

check: $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$ ✓.

Note: $F(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative, $\frac{d}{dx}\left(\frac{1}{3}x^3+4\right) = x^2$.

General antiderivative

Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x) + c$ for some $c \in \mathbb{R}$.

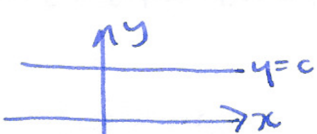
Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = f(x)$ and $G'(x) = f(x)$.

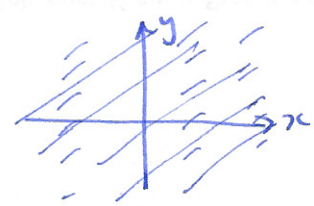
Consider $F(x) - G(x)$, which has derivative $(F(x)-G(x))' = F'(x) - G'(x) = x^2 - x^2 = 0$

so $F(x) - G(x) = c$ constant function. $\square$.

Picture: $f(x)$ gives the slope function for $F(x)$

Examples $f(x) = c$

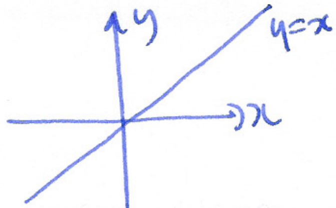




slope $F'(x) = c$ everywhere

$F(x) = cx + d$

$f(x) = x$



$F(x) = \frac{1}{2}x^2 + c$

Examples: find the general antiderivative to $f(x) = \sin(4x)$

guess: $\frac{d}{dx}\left(\cos(4x)\right) = -\sin(4x)\cdot 4$

so $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4}\cdot -\sin(4x)\cdot 4 = \sin(4x)$

so $F(x) = -\frac{1}{4}\cos(4x) + c$

Notation: indefinite integral

$\int f(x)\,dx = F(x) + c$ means: $F(x)+c$ is the general antiderivative for $f(x)$.

Thm $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

Proof $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = \frac{1}{n+1}(n+1)x^n = x^n$ $\square$.

Thm $\int \frac{1}{x}\,dx = \ln|x| + c$

Proof $(x>0)$ $\frac{d}{dx}\left(\ln(x) + c\right) = \frac{1}{x}$.

Thm (sums and constant multiples)

$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int c f(x)\,dx = c\int f(x)\,dx$

warning: no product / quotient / chain rule!

Useful integrals

$\int \sin(x)\,dx = -\cos x + c$

$\int \cos(x)\,dx = \sin x + c$

$\int e^x\,dx = e^x + c$