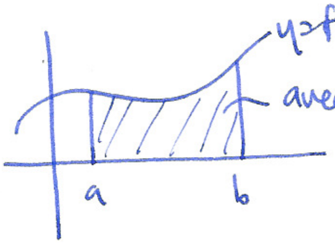
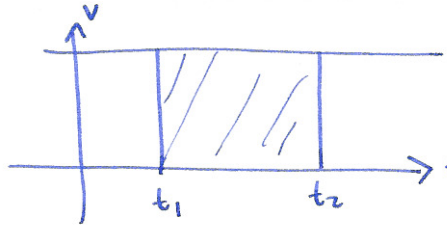
§5.1 Approximating area

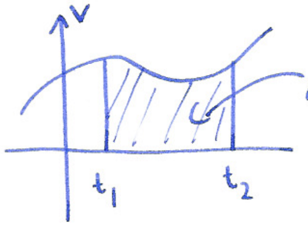


Example



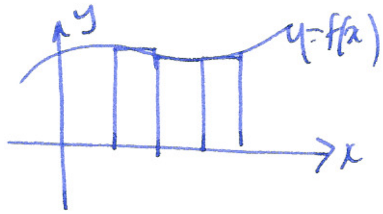
distance travelled = velocity × time
= area under the graph

non-constant velocity:

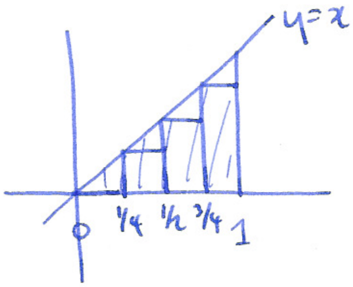


distance travelled = area under the graph

finding the area: approximate by rectangles



Example



find area under graph of $y=x$ between 0 and 1
(answer = 1/2)

approximate with four rectangles: area $\approx$ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$$= \frac{1}{4}\left(f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{1}{4}f(3/4)\right) = \sum_{i=0}^{4} f\left(\frac{i}{4}\right)\frac{1}{4}$$

$$= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f(3/4)\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right) = \frac{1}{4}\left(\frac{6}{4}\right) = \frac{3}{8} = 0.375$$

approximate with n rectangles:

$$f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n-1}{n}\right)\frac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right)$$

$$= \frac{1}{n}\left(f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right) \qquad = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i}{n}$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right)$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + (n-1)\right) \qquad = \sum_{i=0}^{n-1} \frac{1}{n^2} i = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

claim: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

Proof ① induction: assume true for k:

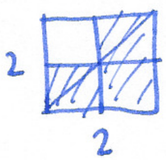
$$s_k = 1+2+\cdots+k = \tfrac{1}{2}k(k+1)$$

$$s_{k+1} = \underbrace{1+2+\cdots+k}_{s_k}+(k+1) = \tfrac{1}{2}k(k+1)+(k+1)$$

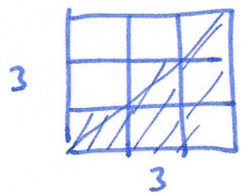
$$= (k+1)\left(\tfrac{1}{2}k+1\right) = \tfrac{1}{2}(k+1)(k+2) \checkmark$$

finally: check $k=1$: $1 = \frac{1}{2}1(2) = 1$ $\square$.

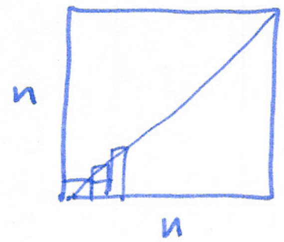
②



$\frac{1}{2}(2)^2+\frac{1}{2}\cdot 2$



$\frac{1}{2}(3)^2+\frac{1}{2}(3)$

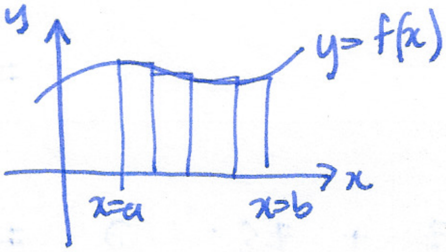


$$\tfrac{1}{2}n^2+\tfrac{1}{2}n$$
$$= \tfrac{1}{2}n(n+1) \ \square$$

so approximate area is $\frac{1}{n^2}\left(1+2+\cdots+(n-1)\right) = \frac{1}{n^2}\frac{1}{2}(n-1)n$

$$= \frac{1}{2}\frac{n^2-n}{n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right) \to \frac{1}{2} \text{ as } n\to\infty.$$

Notation

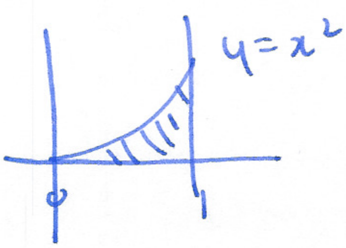


N rectangles of equal width, then $\Delta x = \dfrac{b-a}{N}$

left endpoint rectangles $L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$

right endpoint rectangles $R_N = \sum_{i=1}^{N} f(a+i\Delta x)\Delta x$

midpoint rectangles $M_N = \sum_{i=1}^{N} f\left(a+\left(i-\tfrac{1}{2}\right)\Delta x\right)\Delta x$



$$1+2^2+3^2+\cdots+n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$