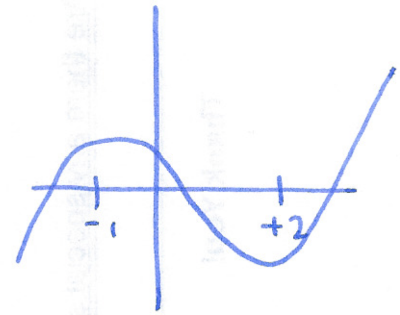
# §4.6 Graph sketching and asymptotes

**Example** sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 = f(x)$




helpful info: • critical points $\quad f'(x) = x^2 - x - 2$

• sign of $f'(x)$

• sign of $f''(x)$ $\quad f''(x) = 2x - 1$

critical points at $x = -1, +2$

$f''(-1) = -3 < 0$ max

$f''(2) = 3 > 0$ min

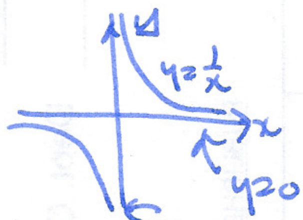
**Example** $f(x) = (4x - x^2)e^x$ $\quad$ critical points $x = 1 \pm \sqrt{5}$

$f'(x) = (4 + 2x - x^2)e^x$ $\quad$ $1+\sqrt{5}$ local max $\quad$ $1-\sqrt{5}$ local min.

$f''(x) = (6 - x^2)e^x$

**Asymptotes**

**Example** $y = \frac{1}{x}$




$y = 0$ horizontal asymptote

$x = 0$ vertical asymptote.

**Def^n** $x = c$ is a vertical asymptote if $\lim_{x \to c^\pm} f(x) = \pm\infty$

$y = c$ is a horizontal asymptote if $\lim_{x \to \pm\infty} f(x) = c$

**Observation**: rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$.

**Example** $f(x) = \frac{x^2 + x + 1}{3x^2 + 2} \sim \frac{x^2}{3x^2} \longrightarrow \frac{1}{3}$ as $x \longrightarrow \infty$.

**Example** sketch graph of $f(x) = \frac{3x + 2}{2x - 4}$

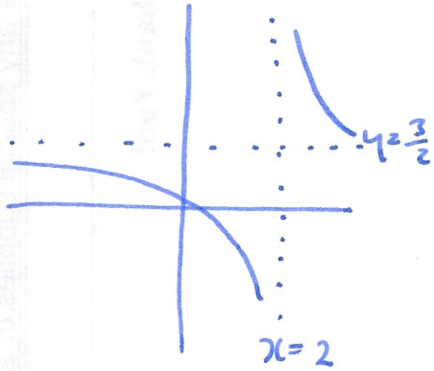
① find vertical asymptotes $\leftrightarrow$ denominator zero: $2x - 4 = 0 \quad x = 2$

② find $f'(x) = \dfrac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \dfrac{-16}{(2x-4)^2} = \dfrac{-4}{(x-2)^2}$ ← always $-ve$ ($f$ decreasing).

no critical points except vertical asymptote at $x=2$.

③ $f''(x) = \dfrac{8}{(x-2)^3}$ $+ve$ $x>2$ concave up

$-ve$ $x<2$ concave down




④ horizontal asymptotes: $\dfrac{3x+2}{2x-4} \sim \dfrac{3x}{2x} = \dfrac{3}{2}$

behaviour near asymptote

| | | $-2/3$ | | $2$ | |
|---|---|---|---|---|---|
| $3x+2$ | $-$ | | $+$ | | $+$ |
| $2x-4$ | $-$ | | $-$ | | $+$ |
| $f(x)$ | $+$ | | $-$ | | $+$ |

Example sketch graph of $f(x) = \dfrac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

① vertical asymptotes: none.

② find $f'(x) = \dfrac{\sqrt{x^2+1} - x\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{x^2+1} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2} > 0$

$\Rightarrow f$ increasing, no critical points

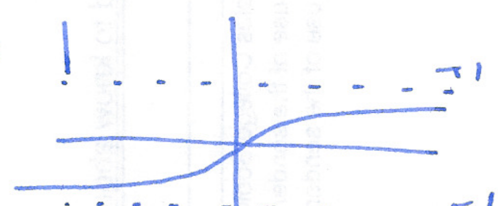
③ $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\cdot 2x$ point of inflection at $x=0$

$+ve$ for $x<0$

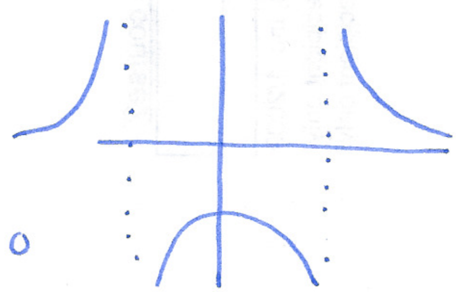
$-ve$ for $x>0$

$\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} \overset{l'H}{=} \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x} = \lim\limits_{x\to\infty} \sqrt{1+1/x^2} = 1$

$\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{-x}{\sqrt{x^2+1}} = -1$




Example $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$



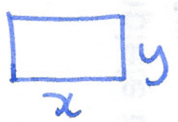
① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2}\cdot 2x$ critical point at $x=0$

3 $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$ etc...

## §4.6 Optimization

<u>Example</u> A piece of wire of length $L$ is bent into a rectangle. What's the largest possible area?




$$\left.\begin{array}{l}\text{area } A = xy \\ \text{length } L = 2x+2y\end{array}\right\} \quad y = \frac{L}{2} - x \qquad A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2.$$

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (max)

so max area of $\frac{L^2}{16}$ occurs when $x = y = \frac{L}{4}$ (square).

<u>Example</u> What shape of cylindrical can minimizes surface area, if you want total volume to be $1\,ft^3$?

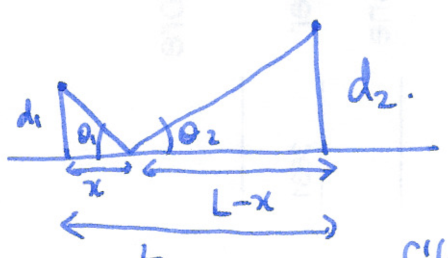

$$V = \pi r^2 h = 1 \Leftrightarrow h = \frac{1}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r h$$

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

$A'(r) = 4\pi r - \frac{2}{r^2}$ solve $A'(r) = 0$: $r^3 = \frac{1}{2\pi}$ $r = \sqrt[3]{1/2\pi}$.

<u>Example</u> a ball bounces off a wall. Show that the path which minimizes distance has equal angle of incidence and reflection.




$$f(x) = \text{distance} = \sqrt{d_1^2+x^2} + \sqrt{d_2^2+(L-x)^2}$$

$$f'(x) = \frac{1}{2}(d_1^2+x^2)^{-1/2} \cdot 2x + \frac{1}{2}\left(d_2^2+(L-x)^2\right)^{-1/2} \cdot 2(L-x)(-1)$$

$f'(x) = 0$: $\dfrac{x}{\sqrt{d_1^2+x^2}} = \dfrac{L-x}{\sqrt{d_2^2+(L-x)^2}} \Leftrightarrow \cos\theta_1 = \cos\theta_2 \Leftrightarrow \theta_1 = \theta_2.$