§4.5 L'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable and $f(a)=g(a)=0$ or $f(a)=g(a)=\pm\infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided this limit exists.

Warning: ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, can't use l'Hôpital on $\lim_{x\to 0}\frac{1}{x}$.

Examples ① $\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{3x^2}{1} = 3$

② $\lim_{x\to \pi/2}\frac{\cos^2 x}{\sin x - 1} = \lim_{x\to\pi/2}\frac{-2\cos x\sin x}{\cos x} = \lim_{x\to\pi/2} -2\sin x = -2$

③ $\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0+}\frac{\ln(x)}{1/x} = \lim_{x\to 0+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0+} -x = 0$

④ $\lim_{x\to 0}\frac{e^x-x-1}{\cos x-1} = \lim_{x\to 0}\frac{e^x-1}{\sin x} = \lim_{x\to 0}\frac{e^x}{-\cos x}$

⑤ $\lim_{x\to 0}\frac{1}{\sin x}-\frac{1}{x} = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{\sin x}{\cos x+\cos x - x\sin x} = 0$

⑥ $\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln(x)}$ note: e^x continuous so $= e^{\lim_{x\to 0+} x\ln(x)}$

$= e^0 = 1$.

Comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$\lim_{x\to\infty}\frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{\sqrt{x}}{4\ln(x)} = \lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{4/x} = \lim_{x\to\infty}\frac{\sqrt{x}}{8} = \infty$

so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial. **Proof** $\lim_{x\to\infty}\frac{e^x}{x^n} = \dots \lim_{x\to\infty}\frac{e^x}{n!} = \infty$.