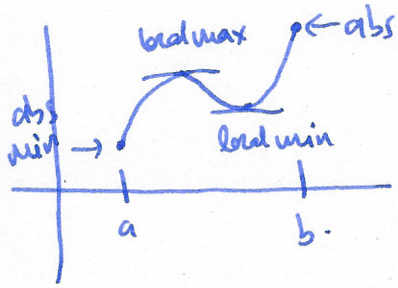
§4.2 Extreme values

suppose $f(x)$ is defined on a closed interval $[a,b]$



Def^n $f(c)$ is the <u>absolute max</u> if $f(c) \geq f(x)$ for all $x \in [a,b]$

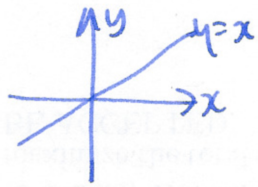
$f(c)$ is the <u>absolute min</u> if $f(c) \leq f(x)$ for all $x \in [a,b]$

<u>note</u>: Q: where is the local/abs max/min ← want x-value

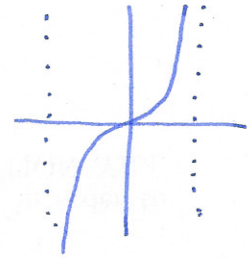
Q: what is the local/abs max/min ← want y-value.

<u>warning</u>: some functions do not have any max or min.

<u>Examples</u>: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$



$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \mapsto \tan(x)$



Th^m If $f(x)$ is continuous on a <u>closed bounded interval</u> then $f(x)$ has both an absolute max and an absolute min.

Def^n $f(x)$ has a <u>local max</u> at $x=c$ if there is a small interval containing c s.t. $f(c)$ is a maximum on this interval.

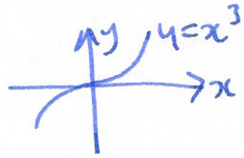
$f(x)$ has a <u>local min</u> at $x=c$ if there is a small interval containing c s.t. $f(c)$ is a minimum on this interval.

Def^n we say that c is a critical point if $f'(c)=0$ or undefined.

Th^m If c is a local min or max, then c is a critical point.

<u>warning</u> $f'(c)=0 \not\Rightarrow$ local max or min.

<u>Example</u> $y=x^3$



$f'(x)=3x^2$ $f'(0)=0$

but $x=0$ not local max or min.

How to find the absolute max or min of a differentiable function on a closed interval $[a,b]$: ① find critical points, evaluate function there.
② check endpoints

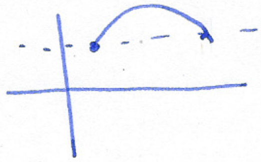
<u>Example</u> find abs max/min of $2x^3-15x^2+24x+7$ on $[0,3]$

examples x^2-8 on $[1,4]$

$\cos(x)\sin(x)$ on $[0,\pi]$.

Thm (Rolle's thm) Suppose $f(x)$ is cts on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is $c \in (a,b)$ s.t. $f'(c)=0$

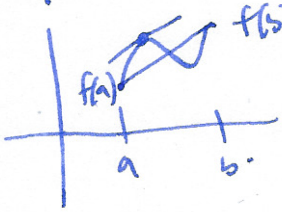
Proof



- if there is a local/max at c, then $f'(c)=0$
- if no local max/min, then $f(x) = \text{const} = f(a)=f(b)$ so $f'(x)=0$ for all $x \in (a,b)$.

§4.3 First derivative test

Thm (Mean value theorem)(MVT) Suppose f is cts on $[a,b]$ and differentiable on (a,b) then there is a $c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$, i.e. there is a point where the slope is equal to the average rate of change.



Corollary If $f(x)$ is differentiable, and $f'(x)=0$, then $f(x)=c$ (constant)

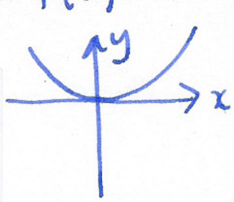
Proof suppose there is a,b, with $f(a)\neq f(b)$. Then $\exists$ $c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a} \neq 0$. $\square$.

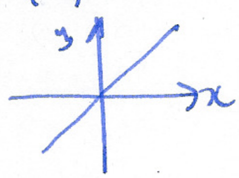
Monotonicity suppose f is differentiable on (a,b):

If $f'(x) > 0$ for all $x \in (a,b)$ then f is increasing on (a,b)

$f'(x) < 0$ decreasing

Example ① $f(x)=x^2$ $\qquad f'(x)=2x$





increasing on $(0,\infty)$

decreasing on $(-\infty,0)$

② $f(x) = x^2-2x-3$

$f'(x) = 2x-2$

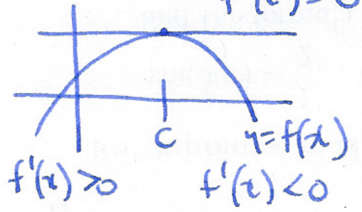
$f'(x) > 0$ where $2x-2>0$

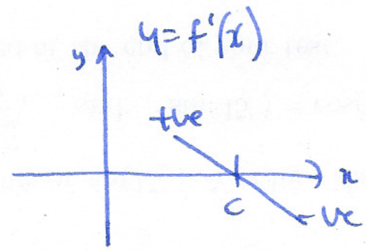
$x>1$

First derivative test

local max:

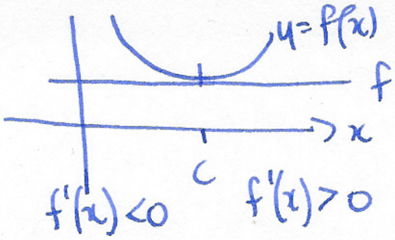
$f'(c)=0$

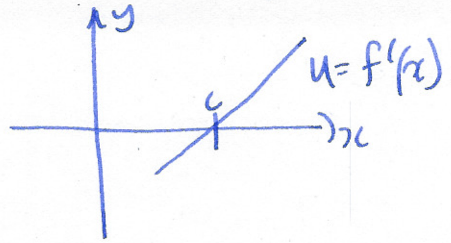




$f'(x)$ goes from positive to negative $\Rightarrow$ local max.

Local min:





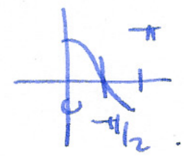
$f'(x)$ goes from negative to positive $\Rightarrow$ local min.

Thm First derivative test If $f(x)$ is differentiable and $f'(c)=0$
then if $f'(x)$ changes from +ve to −ve at $c \Rightarrow c$ local max
−ve to +ve $\Rightarrow c$ local min

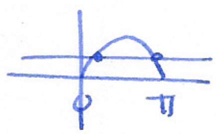
Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0,\pi]$

find critical points: $f'(x) = -2\cos(x)\sin x + \cos x$

solve: $f'(x)=0$ $\quad \cos(x)\left(1-2\sin x\right) = 0 \quad \cos(x) = 0$



$1-\sin 2x = 0 \Leftrightarrow \sin x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$

so critical points are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

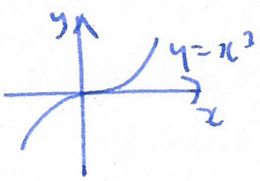
find sign of $f'(x)$ between these:

		$\pi/6$		$\pi/2$		$5\pi/6$	
$\cos(x)$	+		+		−		−
$1-2\sin(x)$	+		−		−		+
$f'(x)$	+		−		+		−

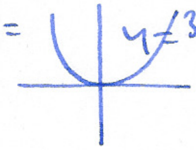
$\Rightarrow \frac{\pi}{6}, \frac{5\pi}{6}$ local max
$\frac{\pi}{2}$ local min

Example critical point not a local max or min.

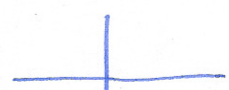
$f(x) = x^3$



$f'(x) =$



$f'(0) = 0$



$f'(x)$ + 0 +

not local max or min.