Proof ~~$y = e^x$~~ ~~$\ln(y) = x$~~

$$y = \ln(x)$$
$$e^y = x$$
$$e^y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \quad \square.$$

Example ① $f(x) = \log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ $\qquad f'(x) = \dfrac{1}{x\ln(b)}$

② $f(x) = x\ln(x)$

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

Hyperbolic trig functions $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\qquad \cosh(x) = \dfrac{e^x + e^{x}}{2}$

$\dfrac{d}{dx}(\sinh(x)) = \cosh x \qquad \dfrac{d}{dx}(\cosh(x)) = \sinh x \qquad \boxed{\cosh^2 x - \sinh^2 x = 1}$

check $(\sinh(x))' = \left(\dfrac{e^x - e^{-x}}{2}\right)' = \dfrac{e^x + e^{-x}}{2} = \cosh(x).$

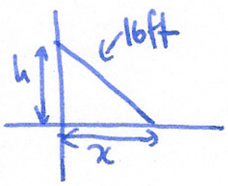
$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ so $\dfrac{d}{dx}(\tanh(x)) = \dfrac{1}{\cosh^2 x}$, etc.

inverse functions: $\dfrac{d}{dx}(\sinh^{-1}(x)) = \dfrac{1}{\sqrt{x^2+1}}$

$\dfrac{d}{dx}(\cosh^{-1}(x)) = \dfrac{1}{\sqrt{x^2-1}} \qquad \dfrac{d}{dx}(\tanh^{-1}(x)) = \dfrac{1}{1-x^2}.$

§3.10 Related rates

Example falling ladder



$x(t)$ distance of foot of ladder from wall

$h(t)$ height of top of ladder against wall.

Q: if $\dfrac{dx}{dt} = 3\text{ft/s}$ what is $\dfrac{dh}{dt}$?

Spse at $t=0$, $x = 5$.

true for **all** t

$$x^2 + h^2 = 16^2$$
$$\frac{dx}{dt} = 3$$

true for **some** t

$$x(0) = 5$$

Example balloon

$V = \frac{4}{3}\pi r^3.$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

$\dfrac{dV}{dt} = 1\text{ft}^3/\text{min}$

$r = 1$ ft.

$1 = 4\pi 1 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{4\pi}$ ft/min

Consider $x^2+h^2=16^2 \leftrightarrow (x(t))^2+(h(t))^2=16^2$

differentiate wrt t : $2x\frac{dx}{dt}+2h\frac{dh}{dt}=0$

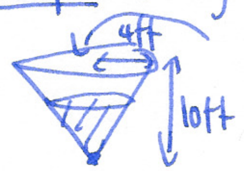
$\frac{dx}{dt}=3$ for all $t \Rightarrow$ $x(t)=3t+5$

$h(t)=\sqrt{16^2-(3t+5)^2}$

so $\frac{dh}{dt}=-\frac{x(t).3}{h(t)}=\frac{-3(3t+5)}{\sqrt{16^2-(3t+5)^2}}$.

hit ground at $5+3t=16$ $t=11/3$ $\frac{dh}{dt}(11/3)=-\frac{16}{\sqrt{16^2-16^2}}$ i.e. $\frac{dh}{dt}\to\infty$ as $t\to 11/3$.

<u>Example</u> filling a conical tank



$10\ ft^3/min$. <u>Q</u>: how fast is the water rising when $h=5ft$?

water in $\frac{dV}{dt}=10\ ft^3/min$

volume of cone: $V=\frac{1}{3}\pi h r^3$

need r in terms of h : $\frac{r}{h}=\frac{4}{10}$ i.e. $r=\frac{2}{5}h$

so $V=\frac{1}{3}\pi h\left(\frac{2}{5}h\right)^2=\frac{4}{75}\pi h^3$

$\frac{dV}{dt}=\frac{12}{75}\pi h^2\frac{dh}{dt}$ so when $h=5$: $10=\frac{12}{75}\pi(5)^2\frac{dh}{dt}$

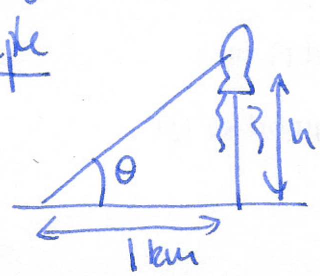
$\frac{dh}{dt}=\frac{10}{4\pi}$ ft/min.

<u>advice</u> ① give things names

② write down relations between things and use implicit differentiation.

③ plug in numbers if necessary.

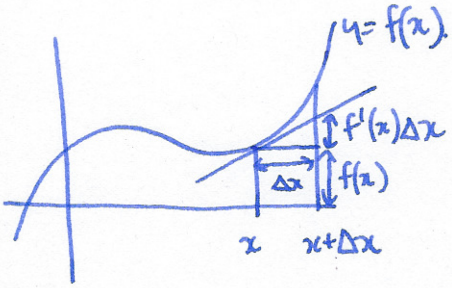
<u>Example</u>



if angle is $\theta=\frac{\pi}{3}$ and rate of change is $\frac{d\theta}{dt}=\frac{1}{2}$ rad/min how fast is the rocket going?

$\frac{h}{1}=\tan\theta$ $\frac{dh}{dt}=\sec^2\theta\frac{d\theta}{dt}$ $\frac{dh}{dt}=10\sec^2\left(\frac{\pi}{3}\right)\frac{1}{2}\approx 1\ km/min$

§4.1 Linear approximation



If $f(x)$ is differentiable at x, and Δx is small, then

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

so change in f is

$$\Delta f \approx f(x+\Delta x) - f(x)$$

$$\Delta f \approx f'(x)\Delta x$$

<u>Example</u> estimate $\sqrt{103}$

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad$ $f(100) = 10$ $\qquad$ so $\Delta f \approx f'(x)\,\Delta x$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad$ $f'(100) = \frac{1}{20}$ $\qquad$ $= \frac{1}{20}\cdot 3$

so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$

<u>Example</u> pizza size: you make an 18" pizza. If you diameter is equal to ± 0.4in, how much pizza do you gain or lose?

$A = \pi r^2 \qquad 2r = D \qquad A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$

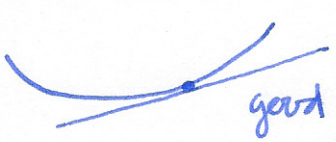
$$A'(D) = \frac{2\pi D}{4} = \frac{\pi D}{2}$$

$\Delta A \approx A'(18)\,\Delta D = \frac{1}{2}\pi . 18 . 0.4 \approx 11 \text{ in}^2.$

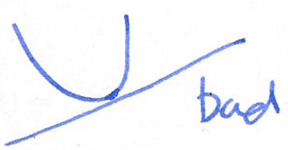
<u>Q</u>: is this good or bad? absolute error $= 11$

percentage error $= \left|\frac{\text{absolute error}}{\text{actual value}}\right| \times 100 = \left(\frac{11}{\pi\, 18^2/4}\right) \times 100$

<u>Observation</u>: when is the linear approximation a good approximation?



$f''(x)$ small



$f''(x)$ large