Examples

1. $e^{4x} = f(g(x))$ where $f(x) = e^x$ $\quad f'(x) = e^x$
$g(x) = 4x$ $\quad g'(x) = 4$

so $(e^{4x})' = f'(g(x)).g'(x) = e^{4x}.4$

2. $\sin^2(x) = f(g(x))$ where $f(x) = x^2$ $\quad f'(x) = 2x$
$g(x) = \sin(x)$ $\quad g'(x) = \cos(x)$

so $(\sin^2(x))' = f'(g(x)).g'(x) = 2\sin(x)\cos(x)$.

3. $\sqrt{x^3+1}$ etc.

Alternative notation $\quad f(g(x)) \leftrightarrow f(u) \quad u = g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\cdot\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}}$$

mnemonic: "cancelling fractions".

Examples $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^3+1}}$.

Proof (of chain rule)

$$\left[f(g(x))\right]' = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}$$

[answer should be $f'(g(x))g'(x)$]

write this as: $\displaystyle\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\cdot\frac{g(x+h) - g(x)}{h}$ $\quad (*)$

set $k = g(x+h) - g(x)$, as g is continuous $h \to 0 \Rightarrow k \to 0$

so $\displaystyle\lim_{k\to 0}\frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x))$

so $(*) = f'(g(x)).g'(x)$ as required $\square$.

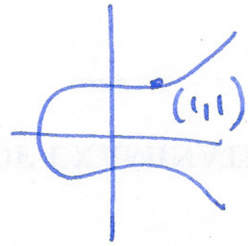
More examples

$$\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}.g'(x)$$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}.g'(x)$$

$$\frac{d}{dx}\big(f(ax+b) \big) = a f'(ax+b)$$

§3.8 Implicit differentiation



Suppose $y^4 + xy = x^3 - x + 2$

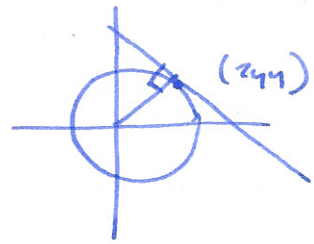
can't solve for y explicitly

$(y(x))^4 + x\,y(x) = x^3 - x + 2$ ← differentiate wrt x using chain rule on $y(x)$.

$$4\,(y(x))^3 \cdot y'(x) + y(x) + x\,y'(x) = 3x^2 - 1.$$

$$y'(x)\,(4y^3 + x) = 3x^2 - 1 - y$$

$$y'(x) = \frac{3x^2 - 1 - y}{4y^3 + x}.$$



Example $x^2 + y^2 = 1$. $\quad 2x + 2yy' = 0 \quad y' = -x/y$

Application: derivatives of inverse functions.

Example

$$y = \ln(x)$$

$$e^y = x$$

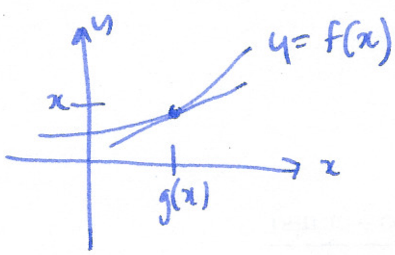
$$e^y \cdot y' = 1$$

$$y' = e^{-y} = e^{-\ln(x)} = \frac{1}{x}.$$

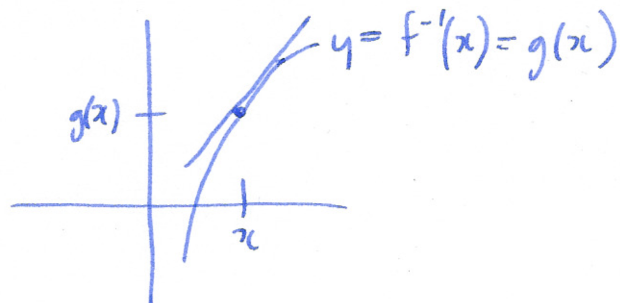
Thm suppose $f(x)$ differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

recall:



$\overrightarrow{\text{reflect in}}$
$y = x$



$\dfrac{1}{\text{slope}}$ at $g(x)$ ⟵⟶ slope at x: $g'(x)$

i.e. $\dfrac{1}{f'(g(x))}$