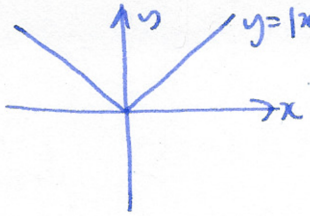
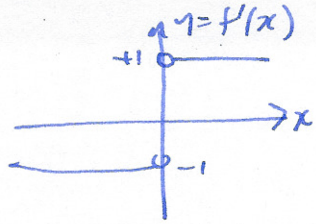
Example $f(x) = |x|$

continuous ✓





claim : not differentiable at $x=0$

check : $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{|x+h|-|x|}{h}$ when $x=0$ $\dfrac{|h|}{h}$ DNE.

$\lim\limits_{h\to 0+} \dfrac{|x+h|-|x|}{h}$ $x=0$, $\lim\limits_{h\to 0+} \dfrac{|h|}{h} = \lim\limits_{h\to 0+} 1 = 1$ $\lim\limits_{h\to 0-} \dfrac{|h|}{h} = \lim\limits_{h\to 0-} \dfrac{-h}{h} = \lim\limits_{h\to 0-} -1 = -1$

$1 \neq -1$ $\lim\limits_{h\to 0} \dfrac{|h|}{h}$ DNE.

Local picture if $f(x)$ is differentiable at $x=c$, then if you look close enough, the graph looks close to a straight line.

Proof (differentiable $\Rightarrow$ cts) $f(x)$ differentiable at $x=c$ means $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists. (want to show $\lim\limits_{h\to 0} f(c+h) = f(c)$)

consider $f(c+h) - f(c) = h\left(\dfrac{f(c+h)-f(c)}{h}\right)$ so $\lim\limits_{h\to 0} f(c+h) - f(c) =$

$\lim\limits_{h\to 0} h \cdot \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = 0 \cdot f'(c) = 0$ □.

§3.3 Product and quotient rules

new functions from old : $f(x)g(x)$ product $\qquad \dfrac{f(x)}{g(x)}$ quotient

Thm (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}\cdot g + f \cdot \frac{dg}{dx}$$

Warning $(fg)' \neq f'g'$!!

Examples ① $\dfrac{d}{dx}(x^2) = \dfrac{d}{dx}(x) \cdot x + x \cdot \dfrac{d}{dx}(x) = 1.x + x.1 = 2x$

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + 3x^2(x^2+1)' = 6x(x^2+1)+3x^2(2x)$

③ $\frac{d}{dx}\left(x^2e^x\right) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

<u>Proof</u> (of product rule) (assume f,g both differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}.$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f(x)g'(x)+g(x)f'(x)$$

$\square$.

<u>Thm</u> <u>Quotient rule</u> (assume f,g differentiable at x, $g(x)\neq 0$)

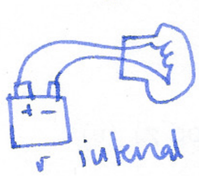
then $\left(\frac{f}{g}\right)'(x) = \frac{gf'-fg'}{g^2} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$

<u>Example</u> ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$

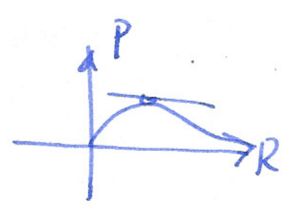
② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)\cdot(e^t)' - (e^t+t)'e^t}{(e^t+t)^2} = \frac{e^t(e^t+t) - (e^t+1)e^t}{(e^t+t)^2}$

$= \frac{te^t - e^t}{(e^t+t)^2}$

③ battery power



power $P = \frac{V^2R}{(r+R)^2}$



<u>Q:</u> when does the battery give maximal power?

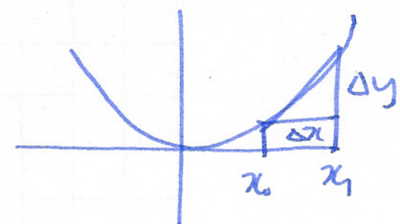
<u>A:</u> when $\frac{dP}{dR}=0$ $\quad P = \frac{V^2R}{(r+R)^2}$ ($P(R)$, V, r constant)

$$\frac{dP}{dR} = \frac{(r+R)^2 (v^2R)' - v^2R\,((r+R)^2)'}{(r+R)^4} = \frac{(r+R)^2 v^2 - v^2R\,(r^2+2rR+R^2)'}{(r+R)^4}$$

$$= \frac{v^2\left[(r+R)^2 - R(2r+2R)\right]}{(r+R)^4} = \frac{v^2(r+R)(r+R-2R)}{(r+R)^4} = \frac{v^2(r-R)}{(r+R)^3} = 0$$

i.e. $R = r$.

§3.4 Rates of change



recall: average rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

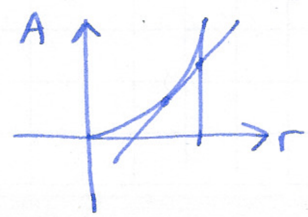
(instantaneous) rate of change $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{x_1\to x_0} \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

observation: if Δx is small, we can use average rate of change to approximate actual rate of change, and vice versa.

Example area of circle is πr^2.

calculate rate of change of area wrt radius: $\dfrac{dA}{dr} = 2\pi r$

e.g. $\left.\dfrac{dA}{dr}\right|_{r=2} = 4\pi \qquad \left.\dfrac{dA}{dr}\right|_{r=5} = 10\pi$



area of ring $\approx 4\pi\,\Delta r$

for small h, $f'(x_0) \approx \dfrac{f(x_0+h)-f(x_0)}{h}$

or $\quad f(x_0+h) \approx f(x_0) + h f'(x_0)$

Example stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$

(s speed in mph) calculate stopping distance when $s = 30$.

$F'(s) = 1.1 + 0.1s \qquad F'(30) = \cancel{4}\,1.1 + 0.1\cdot 30 = 4.1$ ft/mph.

estimate stopping distance at $s = 31$ (using above info):

$F(s+h) \approx F(s) + hF'(s)$

$F(31) \approx F(30) + 1\cdot F'(30) = 78 + 1\times 4.1 = 82.1$