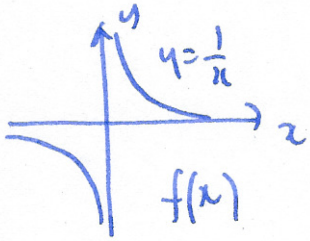
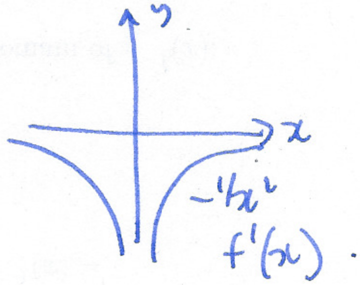
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{x-(x+h)}{(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$





<u>Remarks</u>

① functions : f : domain $\longrightarrow$ range , e.g. $f: \mathbb{R} \to \mathbb{R}$

derivative : (differentiable) functions $\longrightarrow$ functions

$f(x) \longmapsto f'(x)$ (or $\frac{df}{dx}$)

warning: not all functions differentiable!

② "calculus" means rules for doing calculations — we don't have to explicitly compute limits all the time.

<u>Example</u> $f(x) = x^3$ $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2.$

<u>Thm</u> (powers of x) if $f(x) = x^n$, $f'(x) = nx^{n-1}$

(works for all $n \in \mathbb{R}$!)

<u>Examples</u> $\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2 \quad \frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2} \quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$

$\frac{d}{dx}(x^1) = x^0 = 1 \quad \frac{d}{dx}(x^0) = 0.$

<u>Proof</u> Let $f(x) = x^n$, $f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \dots + h^n}_{\text{all of these contain powers of } h \geq h^2.}$

so $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n - x^n}{h} = \lim_{h\to 0} nx^{n-1} + h(\quad)$

$= nx^{n-1}$. $\square$.

<u>Warning</u>: this rule works for polynomials only, not exponentials.

$f(x) = x^{100}$ polynomial $\qquad f(x) = 2^x$ <u>not</u> polynomial

<u>Other useful rules</u>

<u>Thm</u> (linearity) If f and g are differentiable functions, then $f+g$ is differentiable, with $(f+g)' = f' + g'$

$$\leftrightarrow \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

if k is a constant $\qquad (kf)' = kf'$

$$\leftrightarrow \frac{d}{dx}(kf) = k\frac{d}{dx}f.$$

<u>Proof</u> (follows from limit laws)

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

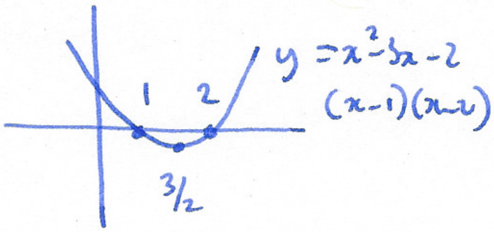
$$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x). \square$$

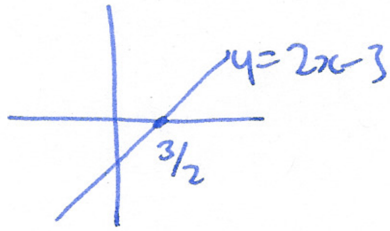
Example $f(x) = x^2 - 3x + 2$. find $f'(x)$.

$$\frac{df}{dx} = \frac{d}{dx}(x^2 - 3x + 2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

$$= \quad 2x \quad - 3 \quad + 0$$

Graphs





Derivative of e^x

consider $f(x) = b^x \quad b > 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} b^x \frac{(b^h - 1)}{h}$$

$$= b^x \underbrace{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

doesn't depend on x!
assume this limit exists and call it m_b.

We have shown: for exponential functions, the derivative is <u>proportional</u> to the value of the function, i.e. $f(x) = b^x$, $f'(x) = m_b b^x$.

in particular, slope at $x = 0$ is $f'(0) = m_b$.

<u>recall</u>: e is defined to be the special number s.t. the slope of e^x at $x = 0$ is equal to 1. Therefore if $\boxed{f(x) = e^x, \ f'(x) = e^x}$

$$\Leftrightarrow \boxed{\frac{d}{dx}(e^x) = e^x}$$

Example $\frac{d}{dx}(7e^x + 4x^2) = 7e^x + 8x$.

<u>Observation</u>: this shows that e^x is <u>not</u> a polynomial!

$\frac{d}{dx}(p(x)) \leftarrow$ degree goes down, eventually zero.

<u>Thm</u> Differentiable $\Rightarrow$ continuous. <u>Warning</u>: continuous $\not\Rightarrow$ differentiable.