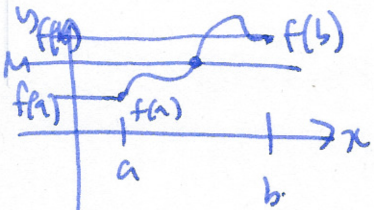
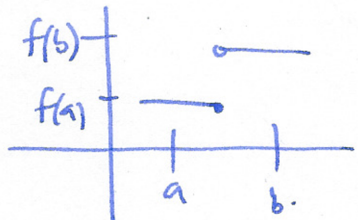
# §2.8 Intermediate Value Theorem (IVT)

"continuous functions can't skip values".




continuous

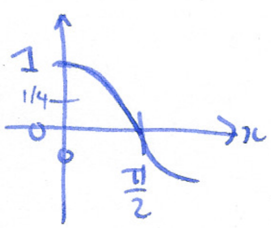



not continuous.

<u>Thm</u> (Intermediate Value Theorem IVT).

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$, then for any number $M$ between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = M$. $\square$

<u>Example</u> show $\cos(x) = \frac{1}{4}$ has at least one solution




consider $[0, \frac{\pi}{2}]$ $\quad \cos(0) = 1$

$\cos(\frac{\pi}{2}) = 0$

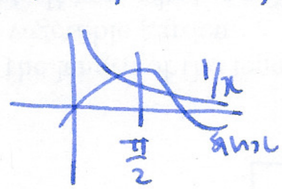
$0 \leq \frac{1}{4} \leq 1 \Rightarrow \exists\, c$ with $\cos(c) = \frac{1}{4}$

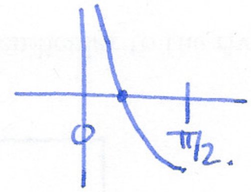
<u>special case: finding zeros</u>:

<u>Corollary</u> if $f(x)$ is continuous on $[a,b]$ and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ with $f(c) = 0$

<u>Bisection method</u>: find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin(x)$



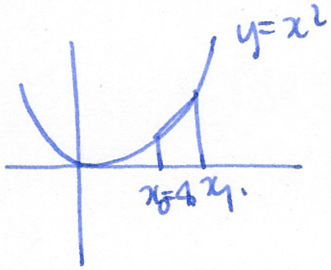

$f(0) = +\infty > 0$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \approx -0.36 < 0$ $\quad$ midpoint: $\frac{\pi}{4}$ $\quad f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566$

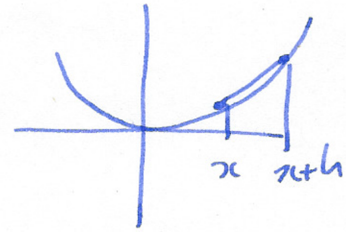
now continue with $[\frac{\pi}{4}, \frac{\pi}{2}]$ etc.

## §3.1 Defn of the derivative

**Recall**: we can compute the average rate of change of a function over an interval




$$[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$




**Q**: how do we compute the slope of the tangent line?

**A idea**: look at average rate of change over small interval $[x,x+h]$ and take limit as $h\to 0$.

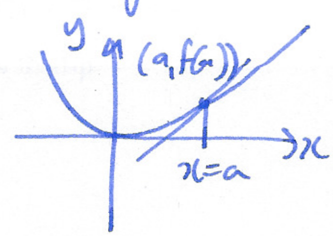
**Defn** the slope of the tangent line at $x=a$ is $\displaystyle\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

**Notation**: also called the <u>derivative</u> written $f'(a)$ (Newton) or $\dfrac{df}{dx}(a)$ (Liebnitz)

if this limit exists, we say the function $f(x)$ is differentiable at $x=a$.

**note**: $\displaystyle\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ same as $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

**Defn** The tangent line to $f(x)$ at the point $(a,f(a))$ is the straight line through $(a,f(a))$ with slope $f'(a)$




the equation for this line is $y-y_0 = m(x-x_0)$

i.e. $y-f(a) = f'(a)(x-a)$ or $y = f(a)+f'(a)(x-a)$

**Example** find tangent line to $y=x^2$ at $x=1$

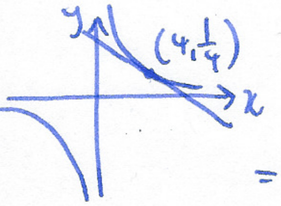
$(x,f(x))$ is $(1,1)$ slope $\displaystyle f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^2-1^2}{h}$

$\displaystyle = \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0} 2+h = 2$

so equation of tangent line is $y-1 = 2(x-1)$ $\quad y = 1+2(x-1)$

$y = 2x-1$.

**Example** find slope of tangent line to $f(x)=\frac{1}{x}$ at $x=4$




slope $f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{\frac{1}{4+h}-\frac{1}{4}}{h}$

$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{4+h}-\frac{1}{4}\right)=\lim_{h\to 0}\frac{1}{h}\,\frac{4-(4+h)}{(4+h)\cdot 4}=\lim_{h\to 0}\frac{4-4-h}{4h(4+h)}$

$=\lim_{h\to 0}\frac{-h}{4h(4+h)}=\lim_{h\to 0}\frac{-1}{4(4+h)}=-\frac{1}{16}$. tangent line $y-\frac{1}{4}=-\frac{1}{16}(x-4)$

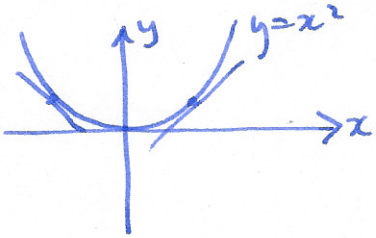
**Example**: straight line $y=mx+b$

find slope at $x=a$: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{m(a+h)+b-(ma+b)}{h}$

$=\lim_{h\to 0}\frac{ma+mh+b-ma-b}{h}=\lim_{h\to 0}\frac{mh}{h}=\lim_{h\to 0}m=m.$

**observation** if $f(x)=b$ (constant), then $f'(x)=0$ for all $x$.

## §3.2 Derivative as a function




$f:\mathbb{R}\to\mathbb{R}$
$x\mapsto x^2$

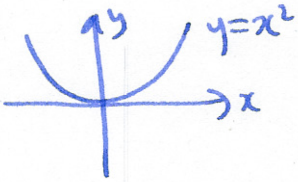
at each input point $x$, there is a tangent line, with a slope, which is a number.

Therefore, we can define a function

$x\mapsto$ slope of tangent line at $x$

**notation**: we call this function $f'(x)$, or "the derivative of $f$".

**Example**



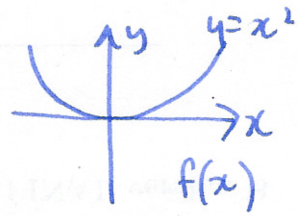

slope at $x$:

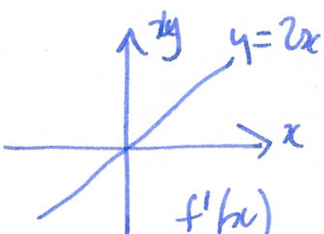
$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

if $f(x)=x^2$ then $f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}2x+h=2x.$

**summary** if $f(x)=x^2$, then $f'(x)=2x$




**Example** $f(x)=\frac{1}{x}$