so $\lim_{x \to c} \left( f(x) + g(x) \right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$ as required $\square$.

**Thm 2** Polynomials are continuous $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

Rational functions $\frac{p(x)}{q(x)}$ are continuous, except where $q(x) = 0$

**Proof** $f(x) = x$ is continuous

so $f(x) . f(x) = x . x = x^2$ is continuous (products)

similarly $x^n$ is continuous

so $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is cts (multiply by a constant; addition)

so $\frac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$ $\square$.

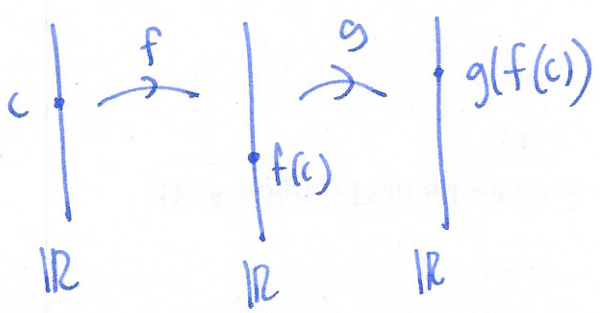
**Useful facts**

**Thm 3**
- $\sin(x)$, $\cos(x)$ are continuous
- $b^x$ is cts.
- $\log_b(x)$ is cts.
- $x^{1/n}$ is cts.

(combinations of these are sometimes called elementary functions)

**Thm 4** (inverse functions) If $f : D \to \mathbb{R}$ is cts, with inverse $f^{-1} : \mathbb{R} \to D$, then $f^{-1}$ is cts.

**Thm 5** (composition) If $f(x)$ is continuous at $x = c$, and $g(x)$ is cts at $x = f(c)$, then $g(f(x))$ is cts at $x = c$




**Examples** $f(x) = \frac{2^x + \sin(x)}{\sqrt{x^2 + x + 1}}$ cts at $x = 1$

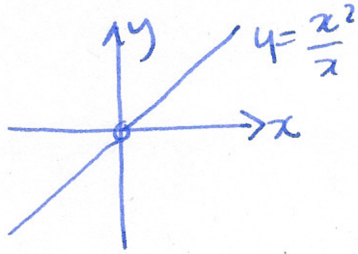
Q: where is $f(x) = \frac{x^2}{\sin(x)}$ cts?

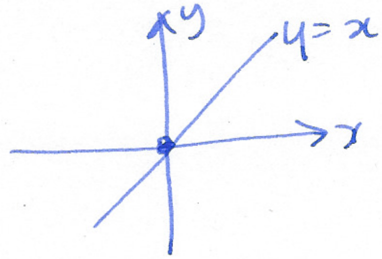
# §2.5 Evaluating limits algebraically

**Example** $\frac{x^2}{x}$ undefined at $x=0$ : $\frac{0}{0}$ interderminate form

but $\lim\limits_{x\to 0} \frac{x^2}{x}$ does not depend on value at $x=0$

$\frac{x^2}{x} = x$ for $x \neq 0$




so $\lim\limits_{x\to 0} \frac{x^2}{x} = \lim\limits_{x\to 0} x = 0$

**Indeterminate forms**: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, $0^0$

**note** $\frac{1}{0}$ not indeterminate, gives limit $\pm\infty$ or DNE.

**Examples** $\lim\limits_{x\to 3} \frac{x^2-4x+3}{x^2+x-12}$ $\quad x=3$: $\frac{9-12+3}{9+3-12} = \frac{0}{0}$

factor: $\frac{(x-3)(x+1)}{(x-3)(x+4)} = \frac{x-1}{x+4}$ $(x\neq 3)$

so $\lim\limits_{x\to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim\limits_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}$

**Example** $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $\quad \frac{2-2}{4-4} = \frac{0}{0}$

$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$ $(x\neq 4)$ so $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$.

**Example** $\lim\limits_{x\to \frac{\pi}{2}} \frac{\tan x}{\sec x}$ $\quad \lim\limits_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1}$

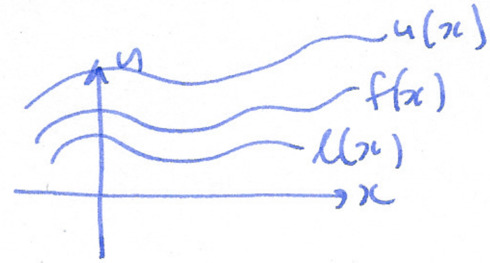
$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{1/\cos x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$$

$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1} \quad (x\neq 1) \quad \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

## §2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin x}{x}$? substitute $x=0$ get $\frac{0}{0}$ undefined.

A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try $x = 0.1, 0.01, \ldots$)

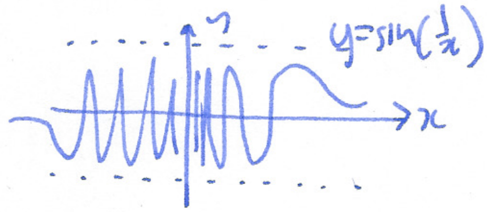


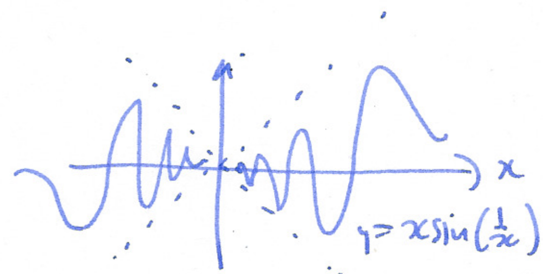

squeeze theorem: suppose $l(x) \leq f(x) \leq u(x)$

Thm suppose $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$

then $\lim_{x\to c} f(x) = L$

Example $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

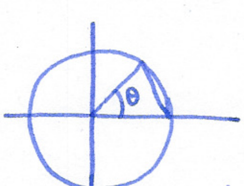


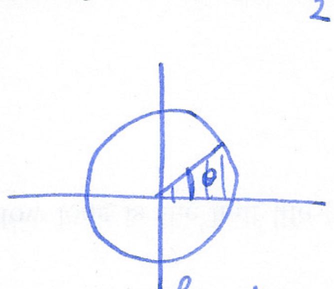



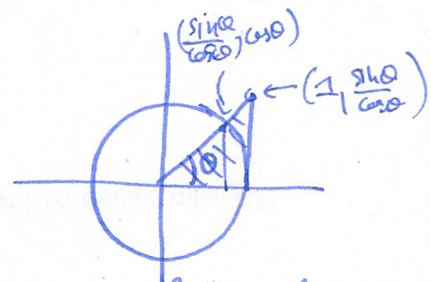

$-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$ $\quad \lim_{x\to 0} |x| = 0 \quad \lim_{x\to 0} -|x| = 0 \Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

Proof (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$. Consider the following three areas:




area of triangle $\leq$ area of sector $\leq$ area of triangle.

$\frac{1}{2}bh$

$$\frac{1}{2}\cdot 1 \cdot \sin\theta \leq \pi r^2 \cdot \frac{\theta}{2\pi} \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\underbrace{\frac{1}{2}\sin\theta \le \overbrace{\frac{1}{2}\theta}^{}}_{\frac{\sin\theta}{\theta}\le 1} \le \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 \qquad \cos\theta \le \frac{\sin\theta}{\theta}$$

So

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1 \qquad \lim_{\theta\to 0} 1 = 1 \qquad \lim_{\theta\to 0}\cos\theta = 1$$

squeeze thm $\Rightarrow$ $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$ $\square$.

Examples ① $\lim_{x\to 0}\frac{\sin 3x}{2x}$ know: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

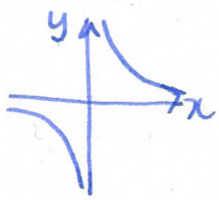
write $3x = \theta$ : $\lim_{\theta\to 0}\frac{\sin\theta}{2(\theta/3)} = \lim_{\theta\to 0}\frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}.$

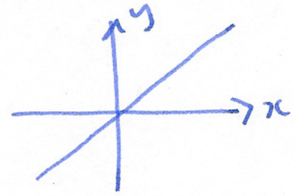
② $\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{t}{\sin t}$ $\qquad \lim_{t\to 0}\frac{\sin t}{t} = 1 \quad \lim_{t\to 0}\frac{1-\cos t}{t} = 0$

$= \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\lim_{t\to 0}\frac{t}{\sin t} = 0\cdot\frac{1}{1} = 0.$

## §2.7 Limits at infinity

key observation: $\lim_{x\to\infty}\frac{1}{x} = 0$




$\lim_{x\to\infty} x = \infty.$

Examples

① $\lim_{x\to\infty}\frac{3x}{2x-1} = \lim_{x\to\infty}\frac{3}{2-1/x} = \frac{3}{2}$

② $\lim_{x\to\infty}\frac{x^2+x}{x-3} = \lim_{x\to\infty}\frac{x+1}{1-3/x} = \infty.$

③ $\lim_{x\to\infty}\frac{1}{x} - \frac{2}{3x+1} = \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{3x+1} = 0-0=0$

④ $\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{4x+1} = \frac{\sqrt{2+1/x^2}}{4+1/x} = \sqrt{2}/4.$