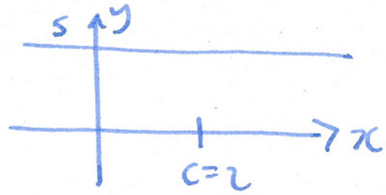
Def^{n} Let f be a function defined on an interval containing c, but not necessarily defined at c. We say "the limit of $f(x)$ as x approaches c is equal to L" if $|f(x)-L|$ becomes arbitrarily small as x gets closer to c.

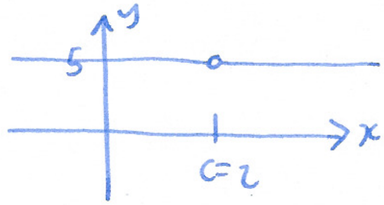
notation: $\lim\limits_{x\to c} f(x) = L$ or $f(x)\to L$ as $x\to c$.

we also say: "$f(x)$ converges to L as x tends to c".

Examples a) $f(x)=5$, $c=2$



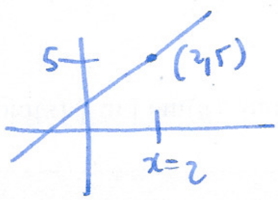
b) $f(x) = \dfrac{5x}{x}$, $c=2$



want to show: $|f(x)-5|$ close to 0 if x close to 2.

$|f(x)-5| = |5-5| = 0$ for all $x\neq 2$, so this is true.

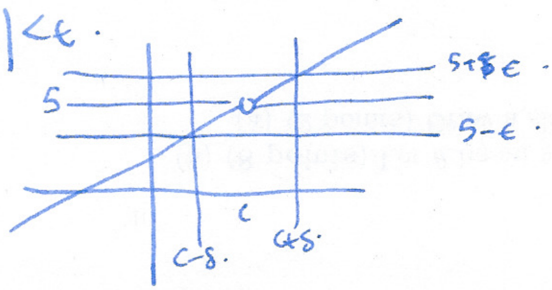
c) $\lim\limits_{x\to 2} 2x+1 = 5$



want to show $|f(x)-5|$ close to 0 when x close to 2.

$|f(x)-5| = |2x+1-5| = |2x-4| = 2|x-2|$ $\quad x$ close to $2 \iff |x-2|$ small.

Precise def^{n} Let $f(x)$ be defined on an interval containing c, but not nec. at c. We say $\lim\limits_{x\to c} f(x)=L$ if for all $\epsilon>0$ there is $\delta>0$ s.t. if $|c-x|<\delta$ then $|f(x)-L|<\epsilon$.



<u>useful facts</u>

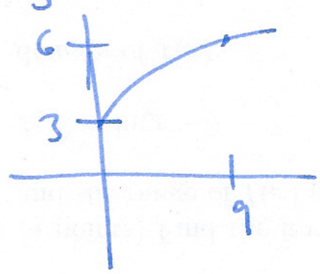
<u>Thm</u> for any constants k, c $\lim_{x \to c} k = k$

$\lim_{x \to c} x = c$

<u>Investigating limits:</u> try
- drawing a picture
- calculating close values
- algebra

<u>Example</u> $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ problem: can't plug in $x=9$, get $\frac{0}{0}$

- draw picture



looks like $f(x) = 6$

- calculate:

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.983
9.1	6.016
8.99	5.998
9.01	6.002

- algebra: difference of two squares

$x - 9 = (\sqrt{x})^2 - 3^2 = (\sqrt{x}-3)(\sqrt{x}+3)$

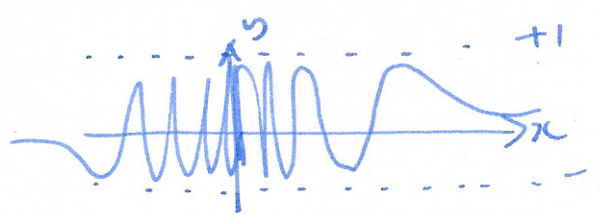
$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \sqrt{x}+3 \quad (x \neq 9).$

$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \sqrt{x}+3 = 6.$

<u>Bad example</u>: no limit

$f(x) = \sin\left(\frac{1}{x}\right)$

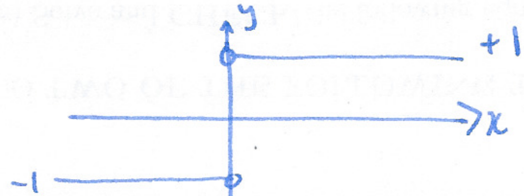
no limit at $x = 0$



<u>note</u>: $f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0$

$f\left(\frac{1}{2\pi n + \frac{\pi}{2}}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$

<u>One sided limits</u>



<u>example</u> $f(x) = \frac{x}{|x|}$.

$f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

sometimes useful to distinguish left limit | right limit | two sided limit.

<u>notation</u> $\lim_{x\to 0+} f(x)$ means right limit (only consider $x>0$)

$\lim_{x\to 0-} f(x)$ means left limit (only consider $x<0$)

<u>note</u>: in order for the (two sided) limit $\lim_{x\to c} f(x)$ to exist, the right limit must equal the left limit: $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)$.

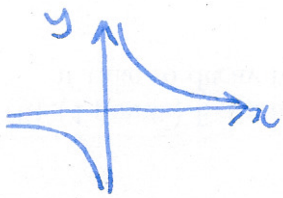
<u>Example</u> $f(x) = \frac{x}{|x|}$ $\left.\begin{array}{l}\lim_{x\to 0+} f(x) = 1 \\ \lim_{x\to 0-} f(x) = -1\end{array}\right\}$ $1 \neq -1$ so $\lim_{x\to 0} f(x)$ DNE.

<u>Infinite limits</u>

we say $\lim_{x\to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large and <u>positive</u> as $x\to c$

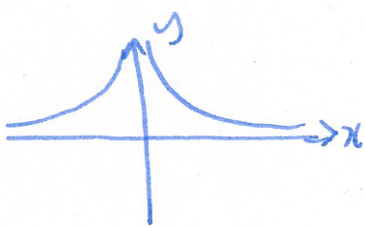
$\lim_{x\to c \cancel{\pm\infty}} f(x) = -\infty$ if $f(x)$ " " <u>negative</u>

<u>Example</u> $f(x) = \frac{1}{x}$



$\left.\begin{array}{l}\lim_{x\to 0+} \frac{1}{x} = +\infty \\ \lim_{x\to 0-} \frac{1}{x} = -\infty\end{array}\right\}$ different! so $\lim_{x\to 0} \frac{1}{x}$ DNE.

<u>Example</u> $f(x) = \frac{1}{x^2}$



$\lim_{x\to 0+} \frac{1}{x^2} = +\infty$ $\lim_{x\to 0-} \frac{1}{x^2} = +\infty$

so $\lim_{x\to 0} \frac{1}{x^2} = +\infty$.

<u>§ 2.3 Basic limit Laws</u>

<u>Example</u> $\lim_{x\to 0} 2x+2 = \lim_{x\to 0} 2x + \lim_{x\to 0} 2 = 0+2 = 2$.

<u>Thm</u> assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist and are finite.

Then:

1) sums: $\lim_{x\to c} \left(f(x)+g(x) \right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ $\quad$ k constant (does not depend on x)

3) products: $\lim_{x\to c} \left(f(x)g(x) \right) = \left(\lim_{x\to c} f(x) \right) \left(\lim_{x\to c} g(x) \right)$

4) quotients: $\lim_{x\to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ $\quad$ as long as $\lim_{x\to c} g(x) \neq 0$

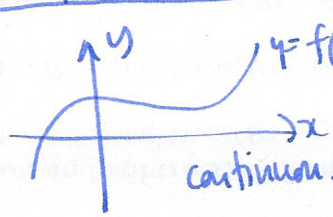
<u>Warning</u>: these rules don't work if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ DNE.

<u>Examples</u>: $\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x \right) \left(\lim_{x\to 3} x \right) = 3.3 = 9$.

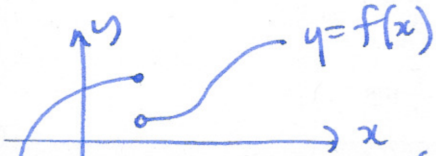
$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 3t} = \frac{7}{6}$$

§2.4 Limits and continuity

<u>Example</u>



continuous

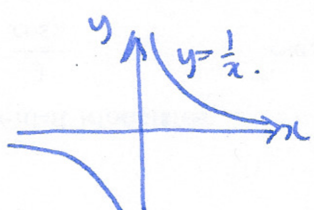


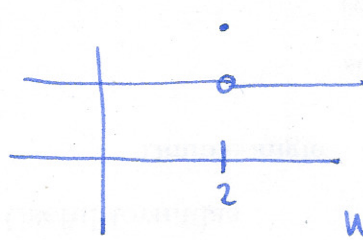
not continuous (discontinuous)

<u>Defn</u> we say $f(x)$ is <u>continuous</u> at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

if the limit does not exist / is infinite / does not equal $f(c)$, then $f(x)$ is not continuous at $x=c$.

<u>Examples</u>

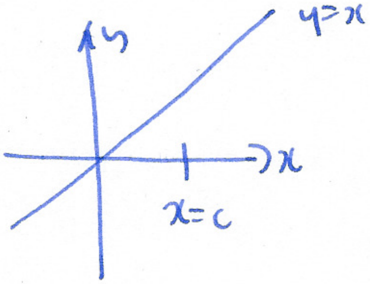




$$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2 \end{cases}$$

not continuous at $x = 2$.

Example show $f(x)=x$ is continuous



$f(c)=c$

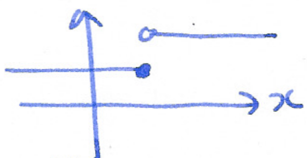
want to show $\lim_{x\to c} f(x) = f(c)$

follows from limit law: $\lim_{x\to c} x = c$

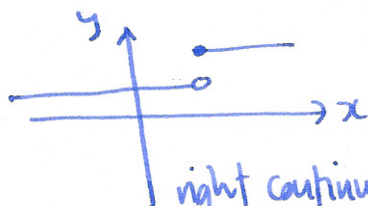
Def^n^ $f(x)$ is <u>left continuous</u> at $x=c$ if $\lim_{x\to c-} f(x) = f(c)$

<u>right continuous</u> $\lim_{x\to c+} f(x) = f(c)$

Examples



left continuous



right continuous

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$.

Building continuous functions

Thm 0 $f(x)=k$ $f(x)=x$ are continuous.

Thm 1 suppose that $f(x)$ and $g(x)$ are both cts at $x=c$. Then the following functions are continuous at $x=c$:

1) $f(x)+g(x)$
2) $kf(x)$ for any constant k
3) $f(x)g(x)$
4) $\dfrac{f(x)}{g(x)}$ if $g(c)\neq 0$

Proof: these follow directly from the limit laws

check 1) $f(x)$ cts means $\lim_{x\to c} f(x) = f(c)$

$g(x)$ cts means $\lim_{x\to c} g(x) = g(c)$