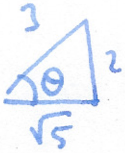
special case: (shift) $\sin(x+\frac{\pi}{2}) = \cos(x)$

Example suppose $\sin\theta = 2/3$ find $\cos\theta$, $\tan\theta$, $\sin 2\theta$



$\cos\theta = \frac{\sqrt{5}}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$ $\quad \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$.

§2.5 Inverse functions

recall: $f: X \to Y$ (domain, range), $x \mapsto f(x)$

want: the inverse function should be the reverse of this $X \leftarrow Y : f^{-1}$, $x \leftarrow\!\shortmid f(x)$

problem: the inverse is often not a function

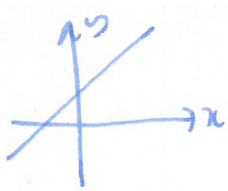
$a \searrow, b \nearrow f(a) = f(b)$ $\quad$ spse $a \neq b$ but $f(a) = f(b)$, what is $f^{-1}(f(a))$?

Q: when does a function have an inverse?

A: when it passes the <u>horizontal line test</u> (one-to-one / injective)

$\Leftrightarrow$ for each value $c \in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$

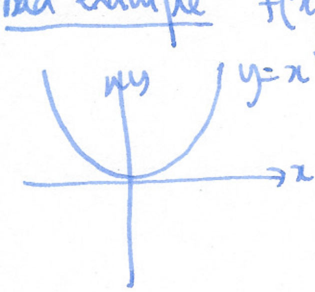
Example $y = x+1$



Q: how do we find a formula for the inverse?

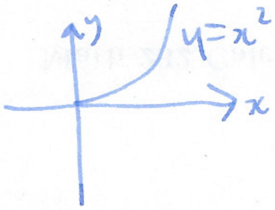
A:
1. write down $y = f(x)$
2. solve for x in term of y i.e. $x = g(y)$
3. $f^{-1}(x) = g(x)$
4. check!

bad example $f(x) = x^2$



problem: doesn't pass horizontal line test

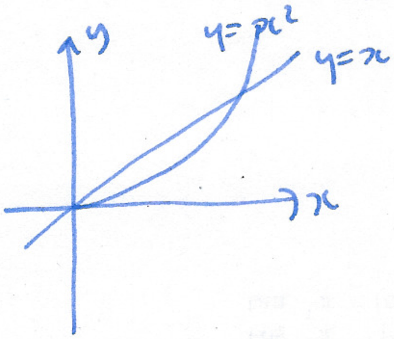
fix: restrict domain, consider $f: [0,\infty) \to [0,\infty)$, $x \mapsto x^2$



<u>does</u> pass horizontal line test so has an inverse we call $\sqrt{x}$

$f^{-1}: (0,\infty) \to (0,\infty)$, $x \mapsto \sqrt{x}$

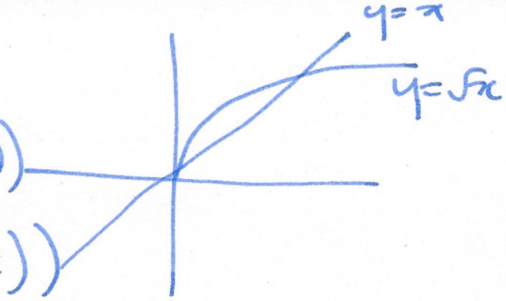
How to draw the graph of the inverse



A: reflect in $y=x$

reason: graph of f: $(x, f(x))$

f^{-1}: $(x, f^{-1}(x))$

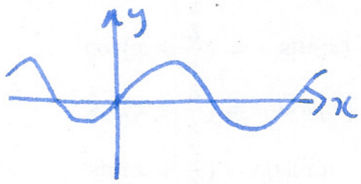


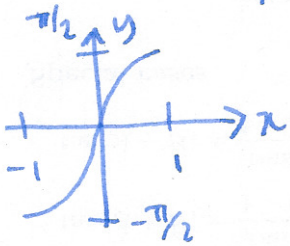
$y=f(x) \Leftrightarrow f^{-1}(y)=x \qquad (x, f(x)) \longleftrightarrow (f^{-1}(y), y)$

swap and relabel.

Inverse trig functions

$y=\sin(x)$





problem: not one-to-one

fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

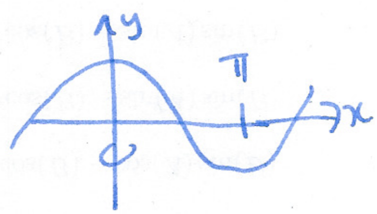
$\sin(x): [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$

$\sin^{-1}(x): [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\parallel$

$\arcsin(x) = \operatorname{asin}(x)$.

similarly $y=\cos(x)$



restrict domain to $[0,\pi]$

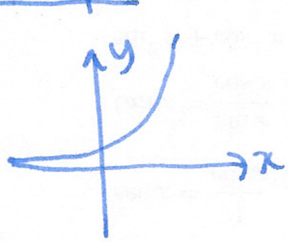
$\cos(x): [0,\pi] \longrightarrow [-1,1]$

$\cos^{-1}(x): [-1,1] \longrightarrow [0,\pi]$

§1.6 Exponential and logarithm functions

Example $x \longmapsto 2^x$

x	-2	1	0	1	2	3
$f(x)$	$1/4$	$1/2$	1	2	4	8



can use any positive number instead of 2 $\quad f(x) = b^x \; (b>0)$

useful properties

- positive $b^x > 0$ for all x
- b^x increasing if $b>1$
  decreasing $0<b<1$
- b^x grows faster than any polynomial x^n

<u>exponent rules</u>: $b^0 = 1$ $\quad b^x b^y = b^{x+y}$ $\quad b^{-x} = \frac{1}{b^x}$ $\quad \frac{b^x}{b^y} = b^{x-y}$

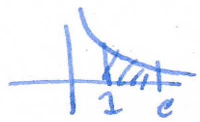
$(b^x)^y = b^{xy}$ $\quad b^{1/y} = \sqrt[y]{b}$

• there is a special exponential function e^x $\quad e = 2.71828\ldots$

<u>key properties</u>

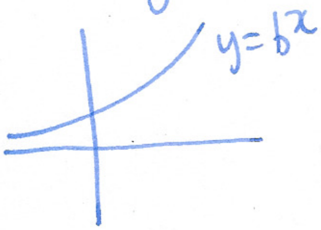
① e is the unique number such that e^x has slope 1 at $x = 0$

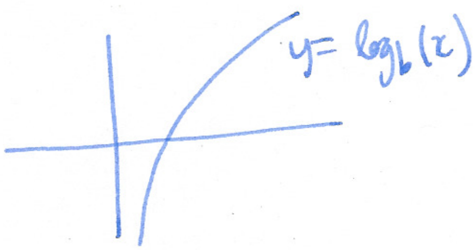
② e is the unique number such that the area under the curve $y = 1/x$ between 1 and e has area 1



<u>logarithms</u>

The logarithm is the <u>inverse function</u> for the exponential function.





the special logarithm with base $b = e$ is called the <u>natural logarithm</u> $\ln(x)$

• inverse function properties: $f^{-1}(f(x)) = x = f(f^{-1}(x))$

$$b^{\log_b(x)} = x = \log_b(b^x)$$

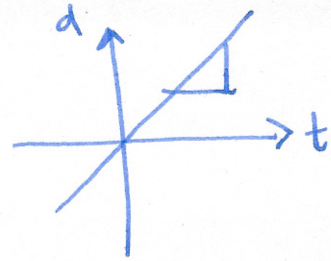
• logarithm rules: $\log_b(1) = 0$ $\quad \log_b(b) = 1$

$\log_b(st) = \log(s) + \log(t)$ $\quad \log_b(1/t) = -\log_b(t)$ $\quad \log_b(s/t) = \log_b(s) - \log_b(t)$

$\log_b(s^t) = t\log_b(s)$

convert between bases: $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ for any a, so $= \dfrac{\ln(x)}{\ln(b)}$.

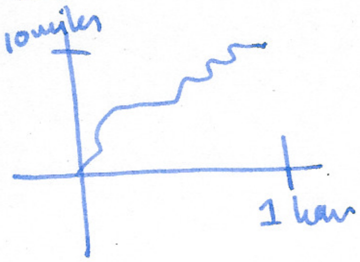
§2.1 Limits, rates of change, tangent lines



motivation: velocity example: driving at constant speed

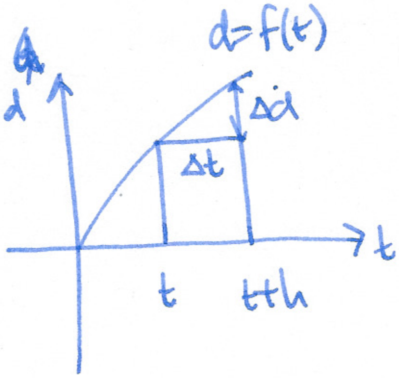
velocity $= \dfrac{\text{distance}}{\text{time}}$ = slope of line.

problem: what happens if you don't travel at constant speed?



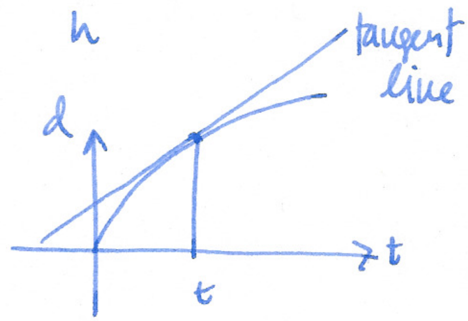
average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

we can look at average speed over any time interval, including very short ones.



average speed on interval $[t, t+h]$

is $\dfrac{\Delta d}{\Delta t} = \dfrac{f(t+h) - f(t)}{(t+h) - t} = \dfrac{f(t+h) - f(t)}{h}$



Q: what is the speed at time t?
(sometimes called the instantaneous speed / rate of change)

A: speed is the slope of the tangent line at t

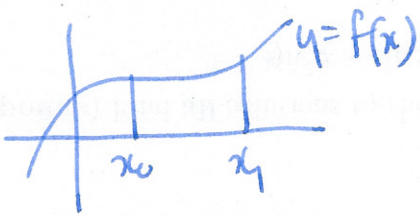
idea (hope): as the length of the ~~t~~ interval $[t, t+h]$ gets small, the average speed gets close to the slope of the tangent line.

This works for "nice" functions.

observation: this works for any function $y = f(x)$, not just speed.

summary average rate of change over an interval $[x_0, x_1]$ is

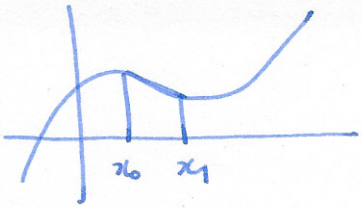
$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



§2.2 Limits

aim: want to find slope of tangent lines

know: average rate of change $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$



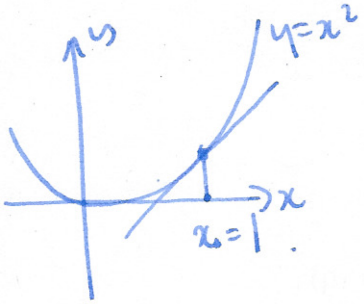
Q: why can't we just set $x_1 = x_0$?

A: doesn't work, get $\dfrac{f(x_0)-f(x_0)}{x_0-x_0} = \dfrac{0}{0}$ undefined.

Observations

① if we draw careful pictures, the average slope gets closer to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculations too:



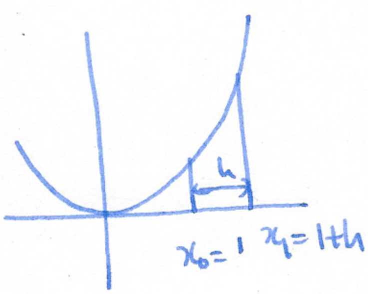
$x_1 = 2$: $\dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3.$

$x_1 = 1.5$: $\dfrac{f(3/2)-f(1)}{3/2-1} = \dfrac{9/4-1}{1/2} = \dfrac{5}{2} = 2.5$

$x_1 = 1.1$: $\dfrac{1.21-1}{0.1} = 2.1$

$x_1 = 1.01$ $\dfrac{1.0201-1}{0.01} = 2.01$

③ seems to work algebraically



average rate of change from $\underset{x_0}{1}$ to $\underset{x_1}{1+h}$

$$= \frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2-1^2}{h} = \frac{1+2h+h^2-1}{h}$$

$$= \frac{2h+h^2}{h} = 2+h$$

$(h \neq 0!)$