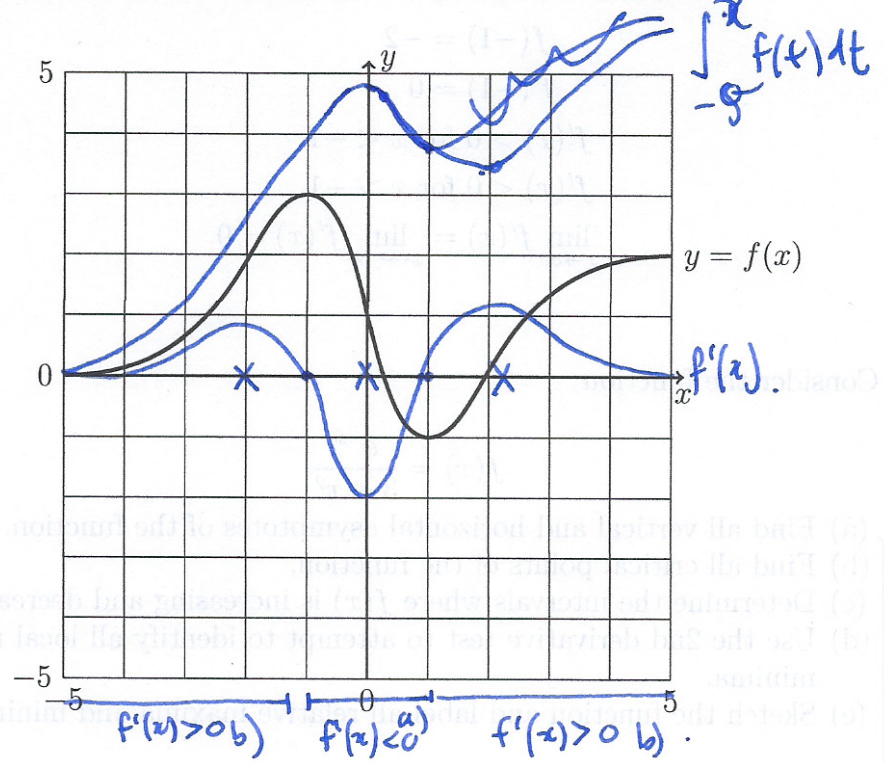
Solutions

1

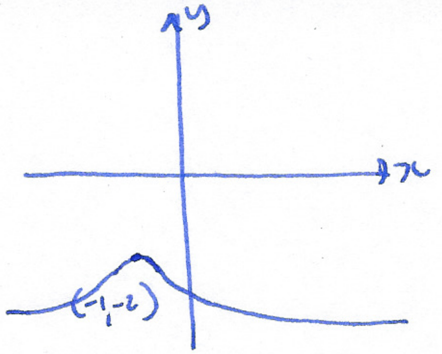
Math 231 Calculus 1 Fall 18 Sample Midterm 3

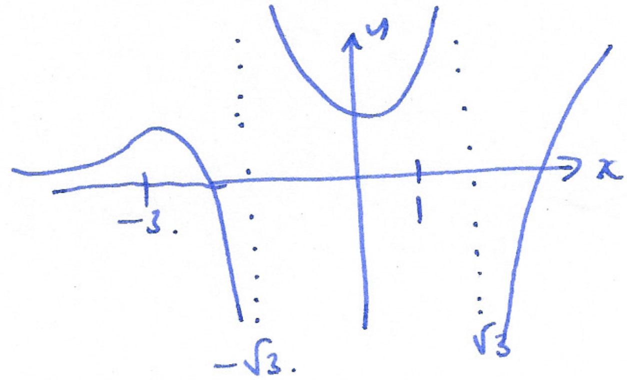
(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$.
(b) Label all regions where $f'(x) > 0$.
(c) What is $\lim_{x\to\infty} f'(x)$? 0
(d) What is $\lim_{x\to-\infty} f''(x)$? 0
(e) Sketch a graph of $f'(x)$ on the figure.
(f) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.
(g) Label the approximate locations of all points of inflection. X on graph

Q2





Q3 $f(x) = \dfrac{e^{-x}}{3-x^2} = \dfrac{e^{-x}}{(x+\sqrt{3})(\sqrt{3}-x)}$

a) vertical asymptotes $x = \pm\sqrt{3}$.

horizontal asymptotes $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{3-x^2} = 0$ $\quad \lim\limits_{x\to-\infty} \dfrac{e^{-x}}{3-x^2} = \infty$.

b) $f'(x) = \dfrac{-(3-x^2)e^{-x} - e^{-x}(-2x)}{(3-x^2)^2} = \dfrac{+e^{-x}(x^2+2x-3)}{(3-x^2)^2} = \dfrac{e^{-x}(x+3)(x-1)}{(3-x^2)^2}$.

critical points $x=-3, x=1$.

c)

		-3		1	
$x+3$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

increasing on $(-\infty,-3)\cup(1,\infty)$

decreasing on $(-3,1)$

d) $f''(x) = \dfrac{(3-x^2)^2\left(-e^{-x}(x^2+2x-3) + e^{-x}(2x+2)\right) - 2(3-x^2)(-2x)e^{-x}(x+3)(x-1)}{(3-x^2)^4}$

$= \dfrac{e^{-x}}{(3-x^2)^4}\Big[(3-x^2)(x+3)(x-1)\underbrace{\left(-(3-x^2)+4x\right)}_{x^2+4x-3} + 2(x+1)(3-x^2)^2\Big]$.

$= \dfrac{e^{-x}(3-x^2)}{(3-x^2)^4}\Big[(x^2+2x-3)(x^2+4x-3) + 2(x+1)\Big]$.

$f''(-3) = \frac{+}{+}(-1)\Big[(\overbrace{9-6-3}^{0})(*) - 4\Big] < 0$ local max

$f''(1) = \frac{+}{+}(+)\Big[\ 0\,(*) + 4\Big] > 0$ local min.

Q4 $g(x) = (x^2-2x)e^x$

a) $g'(x) = (2x-2)e^x + (x^2-2x)e^x = e^x(x^2-2)$

solve $g'(x)=0$ $x = \pm\sqrt{2}$. $(\sqrt{2}, 2(1-\sqrt{2})e^{\sqrt{2}})$ $(-\sqrt{2}, -2(1+\sqrt{2})e^{-\sqrt{2}})$.

b) $f'(x)$ $+$ $-$ $+$ $-\sqrt{2}$ $+\sqrt{2}$. increasing $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$

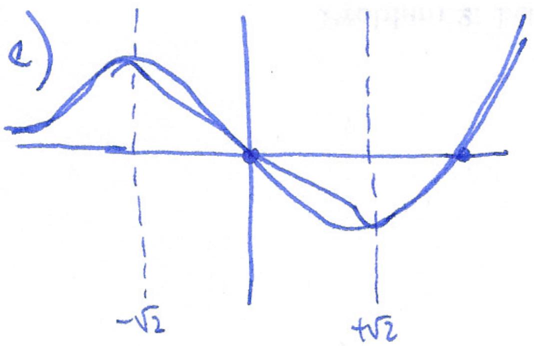
decreasing $(-\sqrt{2},\sqrt{2})$.

c) $g''(x) = e^x(\cancel{2x}\, x^2-2) + e^x(2x) = e^x(x^2+2x-2)$.

$x = \frac{-2\pm\sqrt{4+8}}{2} = -1\pm\sqrt{3}$ points of inflection.

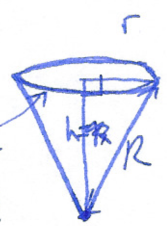
d) concave up $(-\infty,-1-\sqrt{3})\cup(-1+\sqrt{3},\infty)$ concave down $(-1-\sqrt{3},-1+\sqrt{3})$.

e)



Q5 $f'(x) = \frac{1}{e^{-2x}+1} > 0 \Rightarrow$ increasing, max value on $[1,3]$ at 3.

Q6



perimeter $(2\pi-\theta)R = 2\pi r$

$h^2+r^2=R^2$

$V = \frac{1}{3}\pi r^2 h$

$h = \sqrt{R^2-r^2}$

$r = \left(1-\frac{\theta}{2\pi}\right)R$ $\cancel{V = \frac{1}{3}\pi\left(1-\frac{\theta}{2\pi}\right)^2 R^3}$

$$V = \frac{1}{3}\pi\left(1-\frac{\theta}{2\pi}\right)^2 R^2\sqrt{R^2-\left(1-\frac{\theta}{2\pi}\right)^2R^2} = \frac{\pi R^3}{3}\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{1-1+\frac{2\theta}{2\pi}-\frac{\theta^2}{4\pi^2}}.$$

$$= \frac{\pi R^3}{3}\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{\frac{\theta}{\pi}-\frac{\theta^2}{4\pi^2}} = \frac{\pi R^3}{3}\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2}.$$

$$\frac{dV}{d\theta} = \frac{\pi R^3}{3}2\left(1-\frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2} + \frac{\pi R^3}{3}\left(1-\frac{\theta}{2\pi}\right)^2\frac{1}{2}\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right)^{-1/2}\cdot -2\left(1-\frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)$$

$$= \frac{\pi R^3}{3}\left(1-\frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\left[2\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2} - \frac{\left(1-\frac{\theta}{2\pi}\right)^2}{\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2}}\right].$$

$$= \frac{\pi R^3}{3}\,\frac{\left(1-\frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)}{\sqrt{1-\left(1-\theta/2\pi\right)^2}}\left[2\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right) - \left(1-\frac{\theta}{2\pi}\right)^2\right].$$

$$2 - 3\left(1-\frac{\theta}{2\pi}\right)^2$$

$$\frac{dV}{dR} = 0 \Rightarrow \left(1-\frac{\theta}{2\pi}\right)^2 = 2/3.$$

$$1-\frac{\theta}{2\pi} = \sqrt{2}/\sqrt{3}. \qquad \theta = 2\pi\left(1-\sqrt{2}/\sqrt{3}\right)$$

Q7 a) $$\lim_{x\to-\infty}\frac{2+3x}{\sqrt{3x^2-4}} = \lim_{x\to+\infty}\frac{2-3x}{\sqrt{3x^2-4}} \cancel{\overset{l'H}{=} \frac{-3}{\frac{1}{2}(3x^2-4)^{-1/2}\cdot 6x}}$$

$$= \lim_{x\to\infty}\frac{2/x-3}{\sqrt{3-4/x^2}} = \frac{-3}{\sqrt{3}}$$

b) $$\lim_{x\to0+}\tan^{-1}(x)\ln(x) = \lim_{x\to0+}\frac{\ln(x)}{(\tan^{-1}(x))^{-1}} \overset{l'H}{=} \lim_{x\to0+}\frac{1/x}{-(\tan^{-1}(x))^{-2}\cdot\frac{1}{1+x^2}}.$$

$$= -\underbrace{\lim_{x\to0+}(1+x^2)}_{1}\lim_{x\to0+}\frac{(\tan^{-1}(x))^2}{x} \overset{l'H}{=} \lim_{x\to0+}\frac{2(\tan^{-1}(x))\cdot\frac{1}{1+x^2}}{1} = \lim_{x\to0+}2\tan^{-1}(x) = 0$$

c) $$\lim_{x\to0}\frac{1}{x}-\frac{1}{\cot(x)} = \cancel{\lim_{x\to0}\frac{\cot(x)-x}{x\cot(x)}} = \lim_{x\to0}\frac{1}{x}-\tan x = \underbrace{\lim_{x\to0}\frac{1}{x}}_{\text{DNE.}} - \underbrace{\lim_{x\to0}\tan x}_{0}$$

d) $$\lim_{x\to0}\frac{3\sin x-\sin 3x}{\tan^3 2x} \overset{l'H}{=} \lim_{x\to0}\frac{3\cos x-3\cos 3x}{3\tan^2 2x\cdot\sec^2 2x\cdot 2}$$

$$\overset{l'H}{=} \lim_{x\to0}\frac{-3\sin x+9\sin 3x}{6\cdot(2\tan 2x\sec^3 2x\cdot 2+\tan^2 2x\cdot 2\sec 2x\cdot\sec 2x\tan 2x\cdot 2)} = \frac{-3\sin x+9\sin 3x}{24(\tan 2x\sec^3 2x+\tan^3 2x\cdot\sec^2 2x)}$$

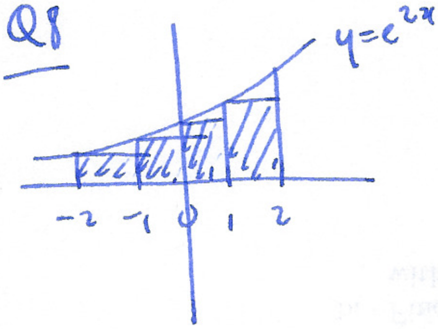
$$= \cancel{l'H}\ \lim_{x\to0}$$

Q4

$$= \lim_{x\to 0} \frac{-3\cos x + 27\cos 3x}{24\left[\sec^4 2x\cdot 2 + \tan 2x\cdot 3\sec^2 2x\cdot \sec 2x\tan 2x\cdot 2 + 3\tan^2 2x\cdot 2\cdot \sec^2 2x + \tan^3 2x\cdot 2\sec 2x\cdot \sec 2x\tan 2x\cdot 2\right]}$$

$$= \frac{-3+27}{24\cdot 2} = -\frac{1}{2}.$$

Q8



R_4 (left) $= (e^{-4}+e^{-2}+e^{0}+e^{2})\cdot 1.$

underestimate.

Q9 a) $\displaystyle\int \frac{1-2x+3x^2}{\sqrt{x}}\,dx = \int x^{-1/2} - 2x^{1/2} + 3x^{3/2}\,dx$

$$= 2x^{1/2} - \frac{2\cdot 2}{3}x^{3/2} + \frac{3\cdot 2}{5}x^{5/2} + c$$

b) $\displaystyle\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx = \left[-\tfrac{1}{2}x^2\right]_{-1}^{0} + \left[\tfrac{1}{2}x^2\right]_{0}^{1}.$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

c) $\displaystyle\int_{1}^{8} 2x^{-1/3}\,dx = \left[3x^{2/3}\right]_{1}^{8} = 3(4-1) = 9$

d) $\displaystyle\int_{0}^{x} \frac{1}{t+2}\,dt = \left[\ln|t+2|\right]_{0}^{x} = \ln(x+2) - \ln(2).$

e) $\displaystyle\int \frac{1}{1+4x^2}\,dx = \int \frac{1}{1+u^2}\frac{dx}{du}\,du = \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\tan^{-1}(u) + c$

$u = 2x$

$\frac{du}{dx} = 2$

$$= \frac{1}{2}\tan^{-1}\left(\overset{2x}{\cancel{\tfrac{x}{2}}}\right) + c.$$

Q10 $v(t) = x'(t) = (t+1)^{-4}$.

$x(t) = -\frac{1}{3}(t+1)^{-3} + c$ $\quad x(0) = 0 \quad -\frac{1}{3} + c = 0 \quad c = \frac{1}{3}$.

$x(t) = \frac{1}{3}\left(1 - \frac{1}{(t+1)^3}\right) \quad \lim_{t \to \infty} \frac{1}{3}\left(1 - \frac{1}{(t+1)^3}\right) = \frac{1}{3}$.

no, $x(t)$ never gets to 1.