Math 231 Calculus 1 Fall 18 Final b

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $x^3 + 3x - 4\sqrt{x}$

$3x^2 + 3 - 2x^{-1/2}$

(b) $x^2 \sin(x)$

$2x\sin(x) + x^2\cos(x)$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{x}{3+\sqrt{x}}$

$$\frac{(3+\sqrt{x}).1 - x\left(\frac{1}{2}x^{-1/2}\right)}{(3+\sqrt{x})^2}$$

(b) $\tan^{-1}(2x)$

$$\frac{1}{1+4x^2} \cdot 2$$

(3) (10 points) Find the derivative of the following functions.

(a) $\ln(\sqrt{\tan(x)})$

$$\frac{1}{\sqrt{\tan(x)}} \cdot \frac{1}{2}(\tan(x))^{-1/2} \cdot \sec^2 x$$

(b) $xy + y^2 = \cos(x)$ (Use implicit differentiation to find y' implicitly.)

$$y + xy' + 2yy' = -\sin(x)$$

$$y' = \frac{-\sin(x) - y}{x + 2y}$$

(4) (10 points) Definition of the derivative as a limit.

(a) State the definition of $f'(x)$ as a limit.

(b) Use the limit definition of the derivative to find the derivative of $f(x) = 1/x$. Do *not* use L'Hôpital's rule.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b) $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h \to 0} \frac{-h}{h(x+h)x}$

$= \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$

(5) (10 points) Consider $f(x) = x^4 - 18x^2 - 4$.

(a) Find the derivative for $f(x)$, and the critical points.

$f'(x) = 4x^3 - 36x = 4x(x^2-9) = 0 \quad x = 0, \pm 3.$

(b) Find the equation of the tangent line at $x = 1$.

$f'(1) = 4 - 36 = -32$

$f(1) = 1 - 18 - 4 = -21$

$y + 21 = -32(x-1)$

(c) Find the intervals for which $f(x)$ is increasing.

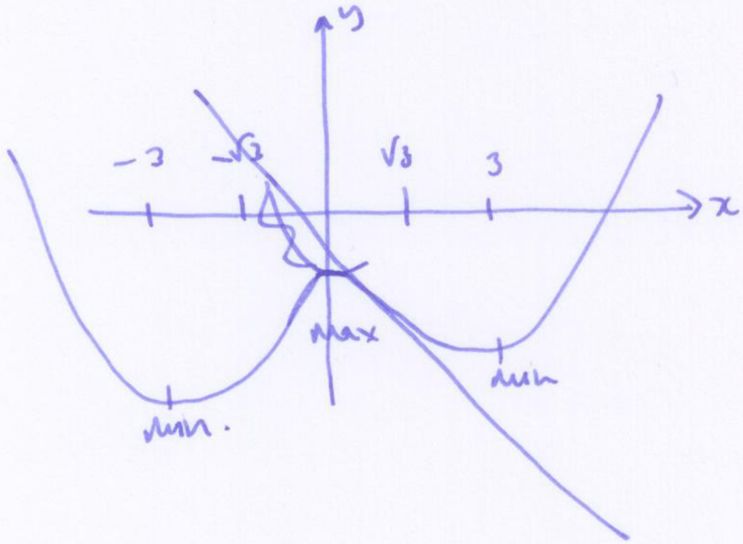
x	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$	$+$
	-3	0	3	
	$-$	$+$	$-$	$+$

$(-3, 0) \cup (3, \infty)$

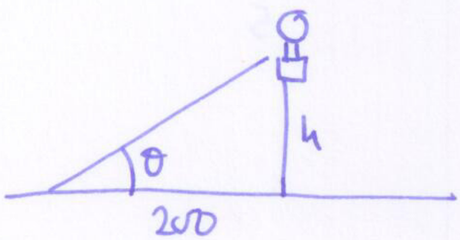
(d) Find the intervals for which $f(x)$ is convex up.

$f''(x) = 12x^2 - 36 = 12(x^2-3) = 0 \quad x = \pm\sqrt{3}. \qquad (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.



(6) (10 points) A hot air ballon rises vertically upwards from a distance of 200m away. If the angle θ at which you see the balloon is increasing at the rate of 0.1 radians/second, how fast is the balloon rising when $\theta = \pi/6$?



$$\tan\theta = \frac{h}{200}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{200}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{200 \cdot 0.1 \cdot}{\cos^2(\pi/6)} = \frac{20}{3/4} = \frac{80}{3}.$$

(7) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$. $\overset{l'H}{=} \lim_{x \to 3} \frac{2x - 2}{2x - 1} = \frac{4}{5}$

(b) Find $\lim_{x \to 0} \frac{\sin 5x}{3x}$. $\overset{l'H}{=} \lim_{x \to 0} \frac{5\cos 5x}{3} = \frac{5}{3}$

(c) Find $\lim_{x \to +\infty} \frac{x^2 + x}{e^{3x}}$. $\overset{l'H}{=} \lim_{x \to \infty} \frac{2x+1}{3e^{3x}} \overset{l'H}{=} \lim_{x \to \infty} \frac{2}{3e^{3x}} = 0$.

(8) (10 points) Evaluate the following integrals.

(a) $\int \left(x^4 + \sqrt[3]{x} + \frac{4}{x} - 3 \right) dx$

$\frac{1}{5}x^5 + \frac{3}{4}x^{4/3} + 4\ln|x| - 3x + c$

(b) $\int_0^{\pi/8} \cos 4x \, dx = \left[\frac{1}{4}\sin(4x) \right]_0^{\pi/8}$

$= \frac{1}{4}\sin(\pi/2) - \frac{1}{4}\sin(0) = \frac{1}{4}.$

(9) (10 points) Evaluate the following integrals.

(a) $\int x \cos(x^2)\, dx$

$u = x^2$

$\frac{du}{dx} = 2x$

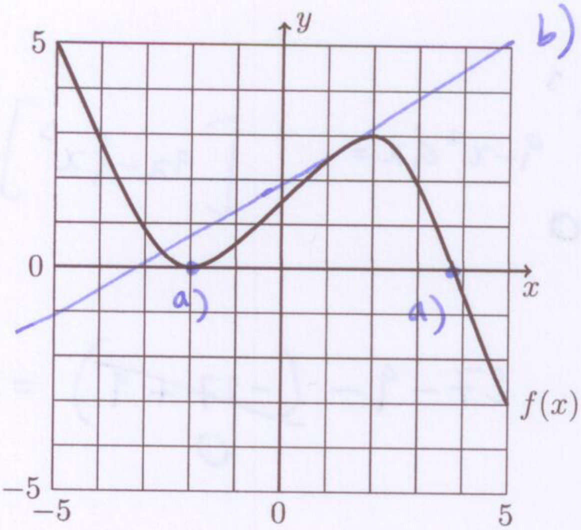
$\int x \cos(u) \frac{dx}{du}\, du = \int x \cos(u) \frac{1}{2x}\, du = \frac{1}{2} \int \cos(u) du$

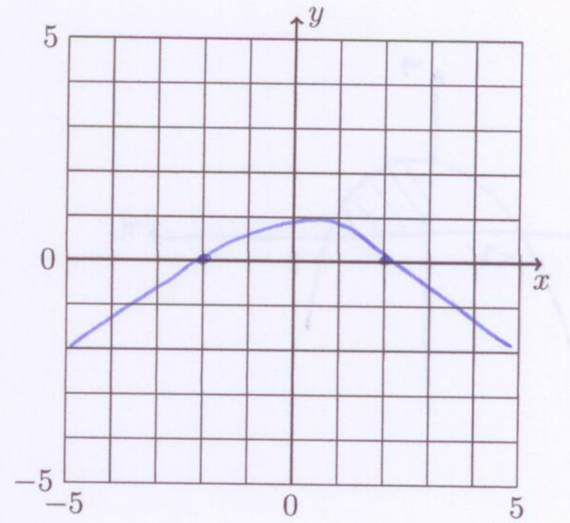
$= \frac{1}{2} \sin(u) + c = \frac{1}{2} \sin(x^2) + c$

(b) If $\int_0^5 f(x)\, dx = 12$ and $\int_2^5 f(x)\, dx = 5$, find $\int_0^2 f(x)\, dx$.

$\int_0^2 f(x)\, dx = \int_0^5 f(x) dx - \int_2^5 f(x) dx = 12 - 5 = 7$

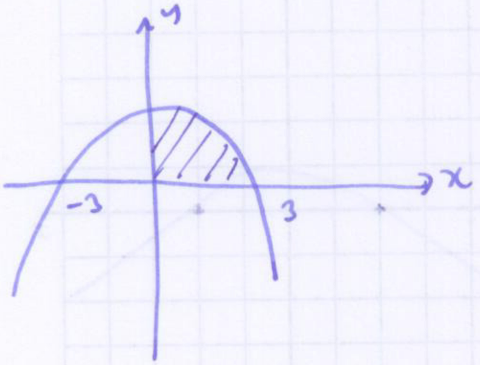
(10) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) Label the roots of $f(x)$ on the graph above.
(b) On the graph above, sketch the tangent line at $x = 1$.
(c) List all the critical points of $f(x)$. $-2, 2$
(d) Sketch $y = f'(x)$ on the right hand graph.
(e) Estimate the intervals where $f(x)$ is concave up. $(-5, 0)$.

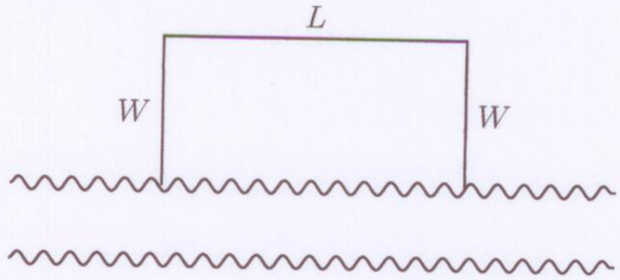
(11) (10 points) Find the area below the graph $f(x) = 9 - x^2$ which lies in the first quadrant.



$$\int_{0}^{3} 9 - x^2 \, dx = \left[9x - \tfrac{1}{3}x^3 \right]_{0}^{3}$$

$$= 27 - 9 - \underbrace{\cancel{(-27 + 9)}}_{0} = 18$$

(12) (10 points) You wish to fence off a rectangular field beside a river and want the area A to be 200 square feet. Using the river as one side, you only need fencing for the other three sides, as shown below. Find the width W and length L that will minimize the fence length.



$LW = A = 200$

$P = L + 2W$

$P = \frac{200}{w} + 2w$

$\frac{dP}{dw} = -\frac{200}{w^2} + 2 = 0$ $w^2 = 100$ $w = 10$

$L = 20$