Math 231 Calculus 1 Fall 18 Final a

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $x^3 - 4x + 3\sqrt{x}$

$3x^2 - 4 + \frac{3}{2}x^{-1/2}$

(b) $x^2 \cos(x)$

$2x\cos(x) - x^2\sin(x)$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{x}{2+\sqrt{x}}$

$$\frac{(2+\sqrt{x}) - (x\frac{1}{2}x^{-1/2}}{(2+\sqrt{x})^2}$$

(b) $\tan^{-1}(3x)$

$$\frac{1}{1+9x^2}\cdot 3$$

(3) (10 points) Find the derivative of the following functions.

(a) $\sqrt{\ln(\tan(x))}$

$$\frac{1}{2}\ln(\tan(x))^{-1/2} \cdot \frac{1}{\tan(x)} \cdot \sec^2(x).$$

(b) $xy - y^2 = \sin(x)$ (Use implicit differentiation to find y' implicitly.)

$$y + xy' - 2yy' = \cos(x)$$

$$y' = \frac{\cos(x) - y}{x - 2y}$$

(4) (10 points) Definition of the derivative as a limit.

(a) State the definition of $f'(x)$ as a limit.

(b) Use the limit definition of the derivative to find the derivative of $f(x) = 1/x$. Do *not* use L'Hôpital's rule.

a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

b) $f'(x) = \lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \dfrac{x\cancel{+h} - (x+h)}{h(x+h)x}$

$= \lim_{h \to 0} \dfrac{-h}{h(x+h)x} = \lim_{h \to 0} \dfrac{-1}{(x+h)(x)} = -\dfrac{1}{x^2}$

(5) (10 points) Consider $f(x) = x^4 - 32x^2 - 2$.

(a) Find the derivative for $f(x)$, and the critical points.

$f'(x) = 4x^3 - 64x = 0 \quad 4x(x^2 - 16) = 0 \quad x = 0, \pm 4$

(b) Find the equation of the tangent line at $x = 1$.

$f'(1) = 4 - 64 = -60 \qquad y + 33 = -60(x - 1)$

$f(1) = 1 - 32 - 2 = -33$

(c) Find the intervals for which $f(x)$ is increasing.

x	$-$	$-$	$+$	$+$
$x-4$	$-$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$	$+$

$- \quad -4 \quad + \quad 0 \quad - \quad 4 \quad +$

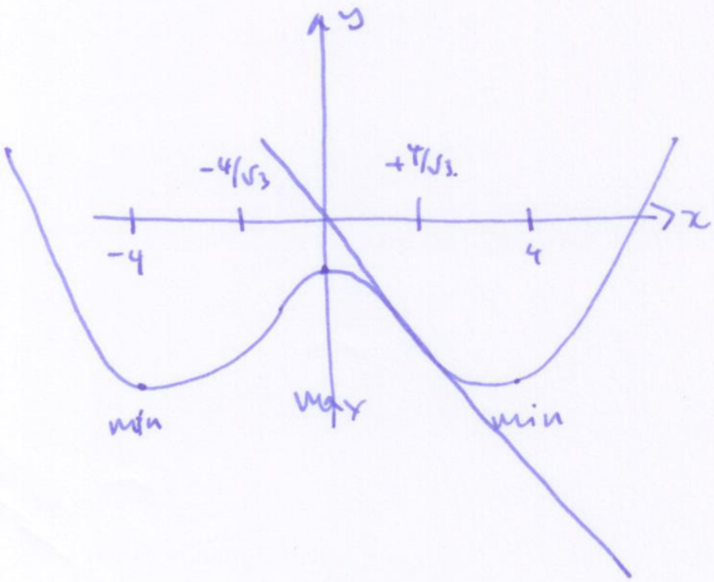
$(-4, 0) \cup (4, \infty)$

(d) Find the intervals for which $f(x)$ is convex up.

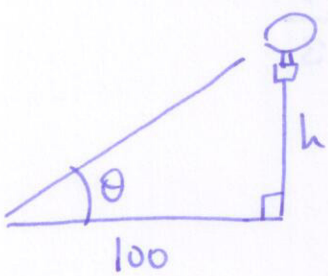
$f''(x) = 12x^2 - 64 = 4(3x^2 - 16) = 0 \qquad x = \pm 4/\sqrt{3}.$

$(-\infty, -4/\sqrt{3}) \cup (4/\sqrt{3}, \infty).$

(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.



(6) (10 points) A hot air ballon rises vertically upwards from a distance of 100m away. If the angle θ at which you see the balloon is increasing at the rate of 0.2 radians/second, how fast is the balloon rising when $\theta = \pi/6$?



$$\frac{h}{100} = \tan\theta$$

$$\frac{1}{100}\frac{dh}{dt} = \sec^2\theta \, \frac{d\theta}{dt}$$

$$\frac{dh}{dt} = \frac{100}{\cos^2(\pi/6)} \cdot 0.2 = \frac{20}{3/4} = \frac{80}{3}.$$

(7) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2}$. $\overset{l'H}{=} \lim_{x\to 2} \frac{2x+1}{2x-1} = \frac{5}{3}$

(b) Find $\lim_{x\to 0} \frac{\sin 3x}{5x}$. $\overset{l'H}{=} \lim_{x\to 0} \frac{3\cos 3x}{5} = \frac{3}{5}$

(c) Find $\lim_{x\to +\infty} \frac{x^2-x}{e^{2x}}$. $= \lim_{x\to \infty} \frac{2x-1}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$

(8) (10 points) Evaluate the following integrals.

(a) $\int \left(x^4 - \sqrt[3]{x} + \frac{3}{x} + 4 \right) dx$

$\frac{1}{5}x^5 - \frac{3x^{4/3}}{4} + 3\ln|x| + 4x + c$

(b) $\int_0^{\pi/6} \sin 3x \, dx$

$\left[-\frac{1}{3}\cos 3x \right]_0^{\pi/6}$

$-\frac{1}{3}\underbrace{\cos(\pi/2)}_{0} + \frac{1}{3}\underbrace{\cos(0)}_{1} = 1/3.$

(9) (10 points) Evaluate the following integrals.

(a) $\int x \sin(x^2)\, dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$\int x \sin(u) \frac{dx}{du}\, du = \int x \sin(u) \frac{1}{2x}\, du = \frac{1}{2} \int \sin(u)\, du$

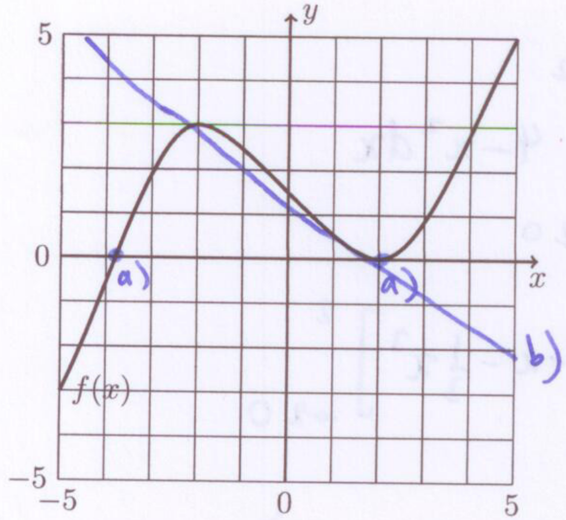
$= -\frac{1}{2} \cos(u) + c = -\frac{1}{2} \cos(x^2) + c$

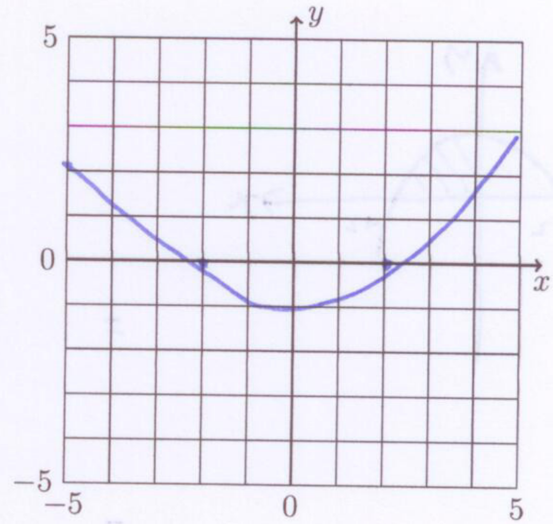
(b) If $\int_0^4 f(x)\, dx = 16$ and $\int_3^4 f(x)\, dx = 4$, find $\int_0^3 f(x)\, dx$.

$\int_0^3 f(x)\,dx = \int_0^4 f(x)\,dx - \int_3^4 f(x)\,dx$

$= 16 - 4 = 12$

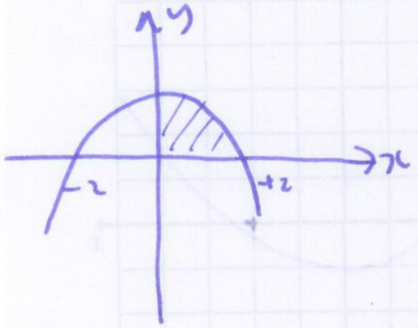
(10) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) Label the roots of $f(x)$ on the graph above.
(b) On the graph above, sketch the tangent line at $x = 1$.
(c) List all the critical points of $f(x)$. $-2, 2$
(d) Sketch $y = f'(x)$ on the right hand graph.
(e) Estimate the intervals where $f(x)$ is concave up. $(0, 5)$.

(11) (10 points) Find the area below the graph $f(x) = 4 - x^2$ which lies in the first quadrant.



$$\int_{\cancel{-2}\,0}^{2} 4-x^2\,dx$$

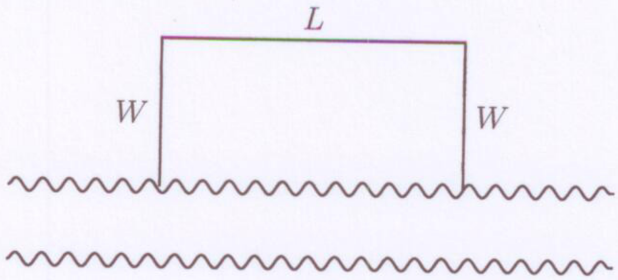
$$= \left[4x-\frac{1}{3}x^3\right]_{\cancel{-2}\,0}^{2}$$

$$= 8-\frac{8}{3}-\cancel{\left(-8+\frac{8}{3}\right)} = \cancel{16\left(1-\frac{1}{3}\right)=\frac{32}{3}}.$$

0

$$8-\frac{8}{3} = \frac{16}{3}$$

(12) (10 points) You wish to fence off a rectangular field beside a river and want the area A to be 800 square feet. Using the river as one side, you only need fencing for the other three sides, as shown below. Find the width W and length L that will minimize the fence length.



$$\left.\begin{aligned} LW &= 800 \\ P &= L + 2W \end{aligned}\right\} \quad P = \frac{800}{W} + 2W$$

$$\frac{dP}{dW} = -\frac{800}{W^2} + 2 = 0$$

$$W^2 = \overset{4}{\cancel{8}}00$$

$$W = 2\cancel{4}0$$

$$L = 40$$