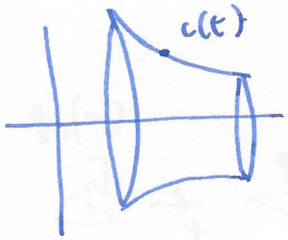
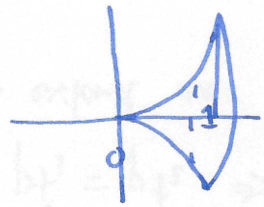
surface area for parameterized curves



$$\text{surface area} = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

example $c(t) = (t, t^3)$



$\frac{dx}{dt} = 1$

$\frac{dy}{dt} = 3t^2$

$$\text{surface area} = 2\pi\int_0^1 \cancel{3}t^3\sqrt{1^2+(3t^2)^2}\,dt = 2\pi\int_0^1 \cancel{3}t^3\sqrt{1+9t^4}\,dt$$

$u = 1+9t^4$

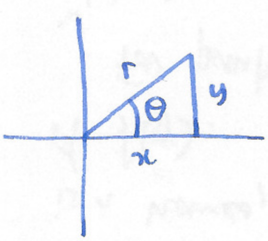
$\frac{du}{dt} = 36t^3$

$$2\pi\int_1^{10} t^3\sqrt{u}\,\frac{dt}{du}\,du = 2\pi\int_1^{10} t^3\sqrt{u}\,\frac{1}{36t^3}\,du = \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du$$

$$= \left[\frac{\pi}{18}\,\frac{2}{3}\,u^{3/2}\right]_1^{10} = \frac{\pi}{27}\left(10^{3/2}-1\right) \approx 3.56$$

example $c(t) = (t - \tanh t, \operatorname{sech} t)$

§11.3 Polar coordinates

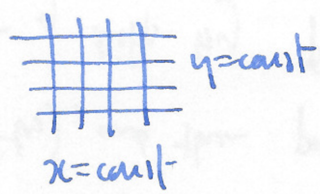


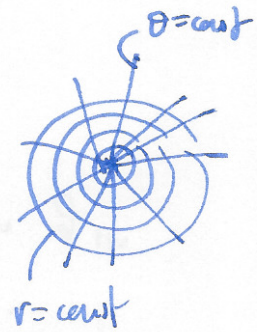
$x = r\cos\theta$

$y = r\sin\theta$

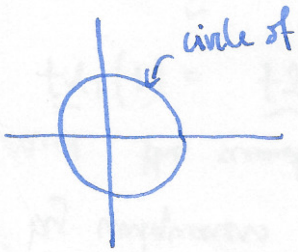
$\tan\theta = y/x$

$r^2 = x^2+y^2$



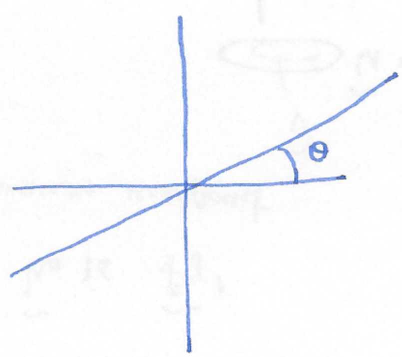


Equations in polar coordinates



circle of radius R: cartesian $x^2+y^2 = R^2$

polar $r = R$



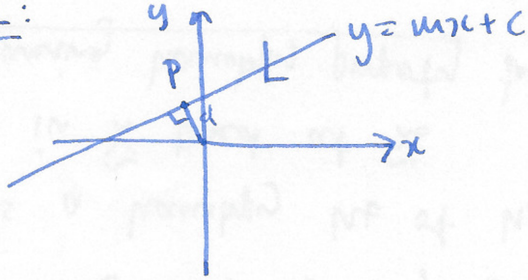
straight line through origin $y = mx$

$\theta = \tan^{-1}(m)$, $\theta = \tan^{-1}(m)+\pi$

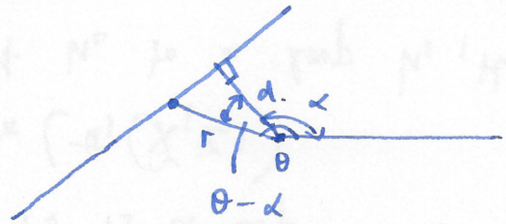
convention: $(-r,\theta) = (r, \theta + \pi)$

so just need $\theta = \tan^{-1}(m)$.

general line:



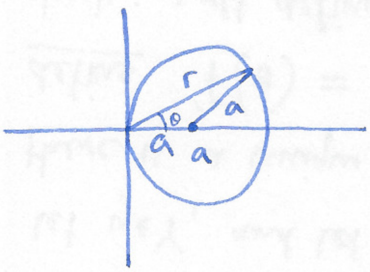
in polars: let P be the closest point on the origin on L, let $d(\underline{o}, P) = d$



$$\frac{d}{r} = \cos(\theta - \alpha)$$

$$r = \frac{d}{\cos(\theta - \alpha)}$$

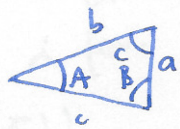
$$= d \sec(\theta - \alpha)$$



cartesian: $(x-a)^2 + y^2 = a^2$

polar: $r = 2a\cos\theta$

cosine rule:

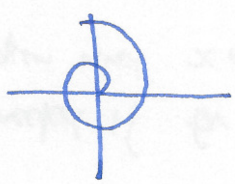


$a^2 = b^2 + c^2 - 2bc\cos\theta$

$$a^2 = r^2 + a^2 - 2ar\cos\theta$$

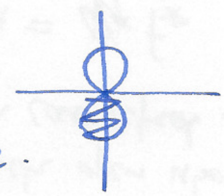
$$r^2 = 2ar\cos\theta \qquad r = 2a\cos\theta$$

Examples sketch $r = \theta$

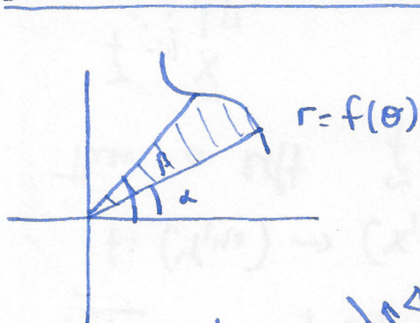


$r = \sin\theta$

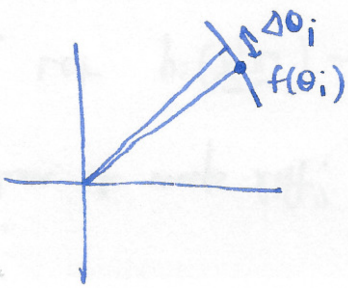
$r = \sin 2\theta$ etc.



§11.4 Area and arc length in polars

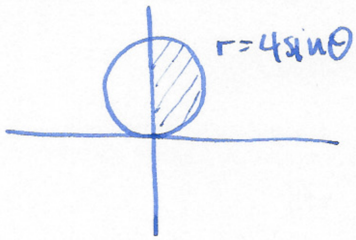


$$\text{area} = \int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$



$$\text{area } \Delta A \approx \frac{\pi r^2}{2\pi/\Delta\theta_i} = \frac{1}{2} r^2 \Delta\theta_i$$

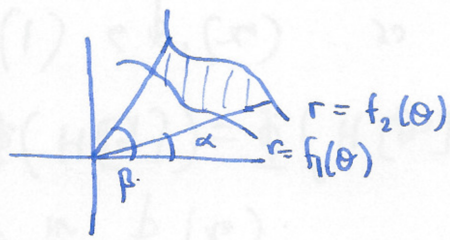
Example



$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(4\sin\theta)^2\,d\theta = \int_0^{\pi/2} 8\sin^2\theta\,d\theta$$

$$= 8\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \Big[4\theta - 2\sin 2\theta\Big]_0^{\pi/2} = 4\cdot\frac{\pi}{2} - 0 = 2\pi$$

area between two curves:



$$\text{area} = \frac{1}{2}\int_\alpha^\beta (f_2(\theta))^2 - (f_1(\theta))^2\,d\theta$$

arc length $r = f(\theta)$ is a parameterized curve with $x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$

so
$$\frac{dx}{d\theta} = f'(\theta)\cos\theta + f(\theta)\cdot -\sin\theta$$
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

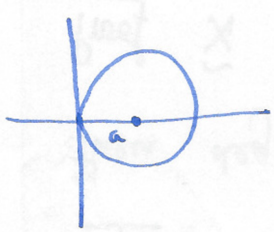
$$\text{arc length } s = \int_\alpha^\beta \sqrt{(f'\cos\theta - f\sin\theta)^2 + (f'\sin\theta + f\cos\theta)^2}\,d\theta$$

$$\left.\begin{array}{l} (f')^2\cos\theta - 2f'f\cos\theta\sin\theta + f^2\sin^2\theta \\ +(f')^2\sin\theta + 2f'f\sin\theta\cos\theta + f^2\cos^2\theta \end{array}\right\} = (f')^2 + f^2$$

so arc length
$$s = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta.$$

Example circle $r = 2a\cos\theta$

$$f(\theta) = 2a\cos\theta$$
$$f'(\theta) = -2a\sin\theta$$



$$\int_0^\pi \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\,d\theta = \int_0^\pi 2a\,d\theta = \Big[2a\theta\Big]_0^\pi = 2a\pi.$$