③ **Fact:** $y = e^x$ is the (unique) solution to $\frac{dy}{dx} = y$ $\quad (f'(x) = f(x))$
(up to multiple)

$$\frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

so $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ radius of convergence $R = \infty$.

How to find a power series solution to a differential equation:

Try $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Example $y' = y$: $a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_1 = a_0$

$2a_2 = a_1 = a_0 \qquad a_2 = \frac{a_0}{2}$

$3a_3 = a_2 \qquad a_3 = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$ so $y = a_0 + \frac{a_0}{1!} x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \cdots$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right).$$

Example $y'' + y' + y = 0$ with $y(0) = 1$, $y'(0) = 1$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad y(0) = a_0 = 1$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \qquad y'(0) = a_1 = 1$

$y'' = 2a_2 + 6a_3 x + \cdots$

$y'' + y' + y = (a_0 + a_1 + 2a_2) + (a_1 + 2a_2 + 6a_3)x + \cdots = 0$

$2a_2 + 2 = 0 \qquad 1 - 2 + 6a_3 = 0 \qquad$ etc.

$a_2 = -1 \qquad a_3 = \frac{1}{6}$

§10.7 Taylor series

suppose $f(x)$ has a power series expansion at $x = c$, i.e.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Q: how do we find the a_i?

A: $f(c) = a_0$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \qquad f'(c) = a_1$

$$f''(x) = 2a_2 + 6a_3(x-c) + \dots \qquad f''(c) = 2a_2$$

etc.

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

Examples $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Thm If $f(x)$ is equal to a power series centered at $x=c$, with radius of convergence $R>0$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where $a_n = \frac{f^{(n)}(c)}{n!}$

Useful facts if the Taylor series converges, we can:

- differentiate
- integrate
- multiply (xe^x)
- substitute (e^{x^2})

Examples xe^x, e^{-x^2}, $e^x \sin(x)$

Fact works for complex numbers $x+iy$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \dots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) = \cos\theta + i\sin\theta$$

How to make l'Hôpital's rule problems:

$$\left.\begin{aligned} \sin(2x) &\approx 2x - \frac{(2x)^3}{3!} + o(x^5) \\ e^{3x} &\approx 1 + 3x + o(x^2) \end{aligned}\right\} \qquad \frac{\sin(2x)}{e^{3x}-1} \approx \frac{2x}{3x} = \frac{2}{3}.$$

Example