Example Q: for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2}\right|$

so converges for $\left|\frac{x}{2}\right| < 1 \iff |x| < 2$ (so $R=2$)

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)^{n+1}}{n+1} \cdot \frac{n}{(x-5)^n}\right| = \lim_{n\to\infty} \left|(x-5)\frac{n}{n+1}\right|$

$= |x-5|$, so converges for $|x-5| < 1$ ($R=1$)

Examples $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad \sum_{n=0} n!\,x^n$

Thm suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R > 0$

then $F(x)$ is differentiable on $(a-R, a+R)$ and furthermore:

$$\left.\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} a_n\, n (x-a)^{n-1} \\ \int F(x)\,dx &= \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n} + C \end{aligned}\right\} \text{ with } \underline{\text{same}} \text{ radius of convergence } R$$

Example ① $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ radius of convergence $R=1$

$$\left.\begin{aligned} &\text{so } \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &\text{and } -\ln|1-x| = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \end{aligned}\right\} \text{radius of convergence } R=1$$

$$\left.\begin{aligned} &② \quad 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2} \\ &\text{so } \tan^{-1}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}\right\} \text{Q: radius of convergence?}$$