Example show $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges (for large n, $a_n \sim \frac{1}{n^2}$)

compare with $b_n = \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \dfrac{\frac{n^2}{n^4-n-1}}{\frac{1}{n^2}} = \frac{n^4}{n^4-n-1} = 1$

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{n^4-n-1}$ by limit comparison test.

Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ converge? compare with $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\frac{1}{\sqrt{n^2-9}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2-9}} = \frac{1}{\sqrt{1-9/n^2}} = 1$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-9}}$ does not converge, by limit comparison test.

§10.4 Absolute and conditional convergence

Q: what about $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ ⊛

Def^n A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

⊛ is absolutely convergent.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ not absolutely convergent.

Thm Absolute convergence $\Rightarrow$ convergence.

Proof $0 \leq a_n + |a_n| \leq 2|a_n|$

$$\underbrace{\sum_{n=1}^{\infty} 2|a_n|}_{\text{conv}} = \underbrace{2\sum_{n=1}^{\infty} |a_n|}_{\text{converges}} \Rightarrow \underbrace{\sum_{n=1}^{\infty} a_n + |a_n|}_{\text{converges}}$$

(by comparison test)

then $\displaystyle\sum_{n=1}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$

converges $\Leftarrow$ converges converges

$\displaystyle \| \sum_{n=1}^{\infty} a_n$ converges $\square$.

<u>Example</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ <u>not</u> absolutely convergent.

<u>Q</u>: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ abs convergent? <u>A</u>: apply integral test to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ <u>no</u>.

<u>Q</u>: does $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ converge?

<u>Defn</u> <u>conditional convergence</u> $\displaystyle\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, but $\displaystyle\sum_{n=1}^{\infty} |a_n|$ does not converge.

<u>Thm</u> Alternating series test

Let (a_n) be decreasing, positive sequence, $a_n \to 0$.

then $\underbrace{\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n}_{S}$ converges. Furthermore: $0 \leq S \leq a_1$

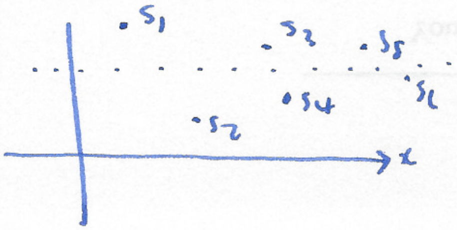
$S_{2n} \leq S \leq S_{2n+1}$ for all n.

<u>Proof</u> even partial sums: $S_{2n} = \underbrace{a_1 - a_2}_{>0} + \underbrace{a_3 - a_4}_{>0} + \dots + \underbrace{a_{2n-1} - a_{2n}}_{>0}$

(positive) increasing sequence

odd partial sums: $S_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{>0} - \underbrace{(a_4 - a_5)}_{>0} - \dots - \underbrace{(a_{2n} - a_{2n+1})}_{>0}$

decreasing sequence.



furthermore $S_{2n} = a_1 - (a_2 - a_3) - \dots - a_{2n}$

so $S_{2n} \leq a_1$ for all n

so S_{2n} is an increasing sequence, bounded above

so $\lim_{n \to \infty} S_{2n}$ exists.

similarly $\lim_{n\to\infty} S_{2n+1}$ exists.

but $\lim_{n\to\infty} S_{2n} - S_{2n+1} = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S_{2n+1}$

$\parallel$

$\lim_{n\to\infty} - a_{2n+1} = 0$ $\square$.

Example show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges (alternating harmonic series)

use: alternating series test: $a_n = \frac{1}{n}$

then a_n is positive decreasing series, and $a_n \to 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$ converges $\square$.

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is conditionally convergent, but not absolutely convergent.

§10.5 Ratio and root tests

fact $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ Q: how do show this converges?

(A: comparison test $n! = 1\cdot 2\cdot 3 \cdots (n-2)(n-1)n > (n-1)^2$

so $\frac{1}{n!} < \frac{1}{(n-1)^2}$).

Thm Ratio test $\{a_n\}$ sequence, and suppose $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists.

then:
1. if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely
2. if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
3. if $\rho = 1$ no information.

Proof if $\rho < 1$, there is a number $\rho < r < 1$, and a number M s.t.

$\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq M$, so $|a_{M+1}| < r|a_M|$

$|a_{M+2}| < r|a_{M+1}| < r^2|a_M|$ etc.

so $\sum_{n=M}^{\infty} |a_n| \leq \sum_{n=M}^{\infty} |a_M| r^n \leq \frac{|a_M|}{1-r}$ so converges by comparison test w/ geometric series.

if $\rho > 1$, then there is $\rho > r > 1$ and M s.t.

$\left|\frac{a_{M+1}}{a_M}\right| > r$ for all $n \geq M$ so $a_n \not\to 0$. $\Rightarrow$ diverges. $\square$.

$|a_{M+n}| \geq r^n |a_M|$

Example ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

② show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \frac{1}{3} = \frac{1}{3} <$

Bad example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} = 1$

Thm (Root test) (a_n) sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

① if $L<1$ then $\sum a_n$ converges absolutely

② if $L>1$ then $\sum a_n$ diverges.

③ if $L=1$ no information

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$.

§10.6 Power series

Def A power series centered at $x=a$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

note: this gives a function of x, if the series converges.

these series always converges for $x=a$! $F(a) = a_0$.

Thm Radius of convergence Let $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then

① $F(x)$ converges only for $x=a$ $(R=0)$

or ② $F(x)$ converges for all x $(R=\infty)$

or ③ there is an $R>0$ s.t. $F(x)$ converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$. It may or may not converge for $x-a=\pm R$.

we call R the radius of convergence.