Example $\underbrace{1}_{\geq 1}+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}_{\geq 1/2}+\cdots$ (harmonic series) (diverges!)

Q: when does a series converge? A: dunno.

Tools:

- **Thm** **Divergence test**: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning: ~~If~~ $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Recall $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$ diverges.

Rules for infinite sums, AKA series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series.

then
$$\sum a_n + \sum b_n = \sum (a_n+b_n)$$
$$\sum a_n - \sum b_n = \sum (a_n-b_n)$$
$$\sum c a_n = c\sum a_n$$

observation
$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=0}^{\infty}\frac{1}{2^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{(\sqrt{2})^{2n}}$$

§10.3 Positive series

positive series: $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots$ all $a_n \geq 0$

note: the sequence S_N is an increasing sequence: $S_{N+1} = S_N + \underbrace{a_{n+1}}_{\geq 0}$

recall: an increasing sequence with an upper bound converges.

Thm Let $S=\sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs:

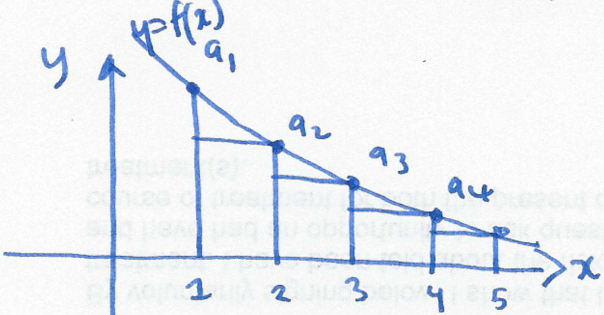
1. the partial sums are bounded above, and S converges.
2. the partial sums are not bounded above, and S diverges.

Thm <u>Integral test</u> Let $a_n = f(n)$ $f(x)$ positive decreasing continuous for $x \geq 1$

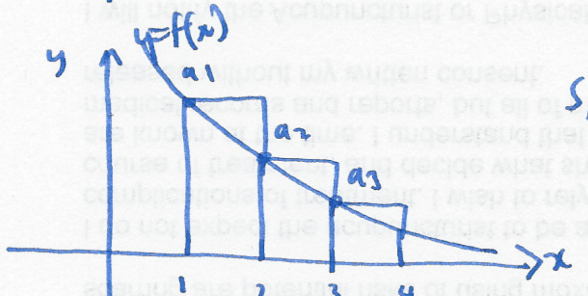
then ① if $\int_1^\infty f(x)dx$ converges $\Rightarrow \sum_{n=1}^\infty a_n$ converges

② if $\int_1^\infty f(x)dx$ diverges $\Rightarrow \sum_{n=1}^\infty a_n$ diverges

Proof



$$S_N - a_1 = a_2 + a_3 + \dots + a_N \leq \int_1^N f(x)dx$$



$$S_N = a_1 + a_2 + \dots + a_N \geq \int_1^N f(x)dx$$

$\square$.

Example $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$ $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\int_1^\infty x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2}\right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty \text{ as } N \to \infty$$

so S diverges.

Thm Convergence of p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

Proof (integral test) $\sum_{n=1}^\infty a_n$ $a_n = f(n)$ $f(x) = \frac{1}{x^p} = x^{-p}$

converges if $p > 1$, diverges if $p \leq 1$ $\square$

Example $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \sum_{n=1}^\infty \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$).

$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots = \sum_{n=1}^\infty \frac{1}{n^3}$ converges (to ?)

Thm Comparison test suppose $0 \leq a_n \leq b_n$ for all $n \geq M$

then ① if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

② if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

Example $\sum_{n=1}^{\infty} 2^{-n^2} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \cdots$

note: $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, ~~series~~ converges.

Thm Limit comparison test a_n, b_n positive sequences

suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$ exists

then: • if $L > 0$ $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges

• if $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof (sketch)

case $L > 0$, then $\frac{a_n}{b_n} \to L$, so $0 < \frac{a_n}{b_n} < R$ $L < R$

$0 < a_n < R b_n$

comparison test: b_n converges $\Rightarrow$ a_n converges.

similarly $\frac{b_n}{a_n} \to \frac{1}{L}$ so $0 < \frac{b_n}{a_n} < R'$ $\frac{1}{L} < R'$

$0 < b_n < R' a_n$

so comparison test: $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ converges.

(if $L=0$ only get one direction) $\square$.