<u>Special case</u>: sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

<u>Thm</u> If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} f(n) = L$

Q: is the converse true?

<u>Example</u> $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim_{x\to\infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n\to\infty} \frac{n-1}{n} = 1$

<u>Example</u> (geometric sequences) $a_n = r^n$

e.g. $2, 4, 8, 16, 32$ $a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $a_n = \frac{1}{3^n}$

$1, 1, 1, \ldots$ $a_n = 1^n$

<u>Fact</u>
$$\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \end{cases}$$

Rules for limits of sequences: same as rules for limits of functions.

Suppose $a_n \to L$ and $b_n \to M$, then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM.$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\left(\lim_{n\to\infty} a_n\right)}{\left(\lim_{n\to\infty} b_n\right)} = \frac{L}{M}$
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ (c constant, i.e. does not depend on n.)

Squeeze Thm if $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} c_n = L$

then $\lim\limits_{n\to\infty} b_n = L$.

Example $\lim\limits_{n\to\infty} \frac{R^n}{n!} = 0$ for any R

Proof there is an integer M s.t. $M \le R \le M+1$

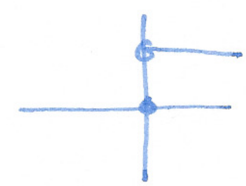
$$0 \le \frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdot \cdots \cdot \frac{R}{M}}_{\text{call this } C}\cdot\frac{R}{M+1}\cdots\cdot\frac{R}{n-1}\cdot\frac{R}{n} \le C\frac{R}{n}$$

so $0 \le \frac{R^n}{n!} \le C\frac{R}{n}$ $\quad \lim\limits_{n\to\infty} 0 = 0 \quad \lim\limits_{n\to\infty} C\frac{R}{n} = 0 \Rightarrow \lim\limits_{n\to\infty} \frac{R^n}{n!} = 0$.

Thm If $f(x)$ is cts and $\lim\limits_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

Important: f continuous. Bad example $f(x) = 0 \quad x \le 0$
$\qquad\qquad 1 \quad x > 0$



then $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) = 1$ for all n and $0 \ne 1$.

$0 = f(0) \ne \lim\limits_{n\to\infty} f(\frac{1}{n}) = 1$.

Example find $\lim\limits_{n\to\infty} e^{n/n+1}$

start with $\lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{1}{1+1/n} = 1$, then $\lim\limits_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$

Def^n A sequence (a_n) is

- bounded above if $a_n \le M$ for all n
- bounded below if $L \le a_n$ for all n
- bounded if $L \le a_n \le M$ for all n.

Thm Convergent subsequences are bounded.

Warning bounded sequences need not converge.

Example $a_n = 1, 0, 1, 0, 1, 0, \ldots$ $\quad a_n = \frac{1-(-1)^n}{2}$.

Thm Bounded monotonic sequences converge

- if (a_n) is increasing and $a_n \leq M$, then $a_n \to l \leq M$.
- if (a_n) is decreasing and $a_n \geq L$, then $a_n \to l \geq L$

Example $a_n = \frac{1}{n}$, decreasing: want $\frac{1}{n} > \frac{1}{n+1}$

$$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$$

lower bound $L = -100$

so $\lim_{n\to\infty} \frac{1}{n} \geq -100$.

Example show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below.

note: $n+1 > n$ so $\sqrt{n+1} > \sqrt{n}$ as $\sqrt{\cdot}$ monotonic $\Rightarrow a_n \geq 0$.

can choose lower bound $L = 0$.

consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{+1/2}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$$

claim $f'(x) < 0$:

$$x+1 > x$$
$$\sqrt{x+1} > \sqrt{x}$$
$$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}} \quad \text{so } f'(x) < 0 \text{ \underline{decreasing}}.$$

§10.2 Series

Defn A series is an infinite sum $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ $\qquad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$1 + 1 + 1 + 1 + \cdots \qquad 1 - 1 + 1 - 1 + 1 - 1 \cdots$

Defn The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

Defn The sum of the infinite series is defined to be the limit of partial sums, if this limit exists. $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

If $\lim_{N\to\infty} S_N = s$, then we say $\sum_{n=1}^{\infty} a_n$ converges and write $\sum_{n=1}^{\infty} a_n = s$.

Example

1. $1+1+1+1\cdots \qquad S_N = \underbrace{1+1+\cdots+1}_{N} = N \qquad \lim_{N\to\infty} N = \infty$

So $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$ does not converge

2. $1-1+1-1+1-1\cdots \qquad s_1 = 1, \; s_2 = 0, \; s_3 = 1, \; s_4 = 0, \ldots$

$(s_n) = 1, 0, 1, 0, 1, 0, \ldots$ does not converge.

Warning non-convergent sums don't work: $(1-1)+(1-1)+(1-1)+\cdots = 0+0+\cdots = 0$

$1-(1-1)-(1-1)-(1-1)\cdots = 1-0-0-0 = 1.$

Geometric series

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$

$s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$

$$S_N = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}$$

$$\frac{1}{2} S_N = \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{N+1}}$$

$$S_N - \frac{1}{2} S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$\frac{1}{2} S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$S_N = 1 - \frac{1}{2^N}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{2^N} = 1 \quad \text{, so} \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

<u>General case</u> $\quad c + cr + cr^2 + cr^3 + \cdots = \sum_{n=0}^{\infty} cr^n$

$$S_N = c + cr + cr^2 + \cdots + cr^N$$

$$rS_N = rc + cr^2 + cr^3 + \cdots + cr^{N+1}$$

$$S_N - rS_N = c - cr^{N+1}$$

$$S_N(1-r) = c - cr^{N+1}$$

$$S_N = \frac{c(1-r^{N+1})}{1-r} \qquad \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{c(1-r^{N+1})}{1-r} = \frac{c}{1-r} \quad \text{if } |r|<1$$

otherwise <u>does not converge</u>.

<u>special case</u> (Telescoping)

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$\text{so} \quad S_N = 1 - \frac{1}{N+1} \qquad \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$