deg 2: $\quad T_2''(x) = 2a_2$

$$T_2''(a) = 2a_2 = f''(a).$$

so $\quad T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$

<u>in general</u> $\quad \cancel{T_n(x) = f(a) + a f'(a)(x-a) +}$

$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

differentiate k times:

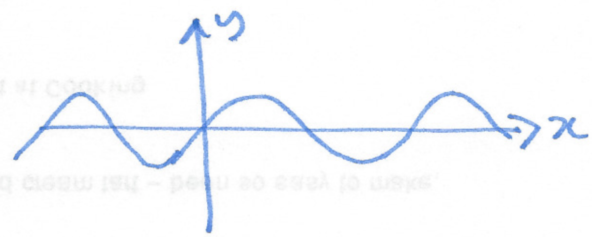
$$T_n^{(k)}(x) = k!\,a_k + (\text{stuff with } (x-a) \text{ factors})$$

$$T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a) \quad \text{so} \quad a_k = \frac{1}{k!} f^{(k)}(a)$$

so $\quad T_n(x) = f(a) + f'(a)(x-a) + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

<u>Example</u> 1 $\quad y = \sin(x)$ at $x = 0$



$f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos(x) \qquad f'(0) = 1$

$f''(x) = -\sin(x) \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$

so $\quad T_n(x) = 0 + 1\cdot x + 0\cdot x^2 + (-1)x^3 + \cdots$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

② $y=\cos(x)$ at $x=0$

$$f(x)=\cos(x) \qquad f(0)=1$$
$$f'(x)=-\sin(x) \qquad f'(0)=0$$
$$f''(x)=-\cos(x) \qquad f''(0)=-1$$
$$f^{(3)}(x)=\sin(x) \qquad f^{(3)}(0)=0$$
$$f^{(4)}(x)=\cos(x) \qquad f^{(4)}(0)=0$$

$$T_n(x) = 1+0x+(-1)\frac{x^2}{2!}+0\frac{x^3}{3!}+\cdots$$
$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

③ $y=e^x$ at $x=0$

$$f(x)=e^x \qquad f(0)=1$$
$$f'(x)=e^x \qquad f'(0)=1$$
$$f''(x)=e^x \qquad f''(0)=1$$

$$T_n(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

<u>Remark</u>
$$e^{i\theta} = 1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\cdots$$
$$= 1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+\cdots$$
$$= \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)$$
$$= \cos\theta + i\sin\theta$$

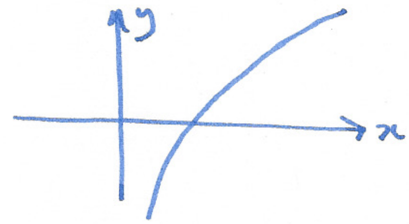
$$\boxed{e^{i\theta}=\cos\theta+i\sin\theta} \qquad e^{i\pi}=\cos(\pi)+i\sin(\pi)=-1$$

$$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$
$$\parallel$$
$$(e^{i\theta})^2 = (\cos\theta+i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

$$\left.\right\} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

hw $e^{i3\theta}$.

<u>Example</u> $y = \ln(x)$ at $x = 1$



$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(1) = -6 = -2.3.$

$f^{(5)}(x) = 4!\, x^{-5}$ $\quad f^{(5)}(1) = 4!$

$f^{(k)}(x) = (k-1)!(-1)^{k+1} x^{-k}$ $\quad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $T_n(x) = 0 + 1(x-1) + (-1)\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} + (-3!)\frac{(x-1)^4}{4!} + \cdots$

$= (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

<u>warning</u>: this may not converge!

$T_n(3) = 2 - \frac{2^2}{2} + \frac{3^3}{3} - \frac{2^4}{4} + \cdots$ does not converge.

<u>Q</u>: how good is the approximation?

<u>Thm</u> (Error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is cts.

Let K be an upper bound on $|f^{(n+1)}(u)|$ for all u in $[a, x]$

then $|T_n(x) - f(x)| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$

<u>Example</u> $f(x) = e^x$ find $T_4(x)$ at $x = 0$, find an error bound for $T_4(1)$

$f(x) = e^x$ $\quad f(0) = 1$ $\qquad T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$\vdots$

$f^{(4)}(x) = e^x$ $\quad f^{(4)}(0) = 1$ $\qquad T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.708\overline{33}.$

error bound: need to find K s.t. $|f^{(n+1)}(u)| \le K$ for all $u \in [0,1]$

note i.e.: $|e^u| \le K$ for $u \in [0,1]$, can choose $K = e$

so $|e - T_4(1)| \le \frac{e \cdot 1^{n+1}}{(n+1)!} = \frac{e}{120}$

§10.1 Sequences

Def: A sequence is a list of numbers indexed by $\mathbb{N}$ = positive integers.

Examples $1, 2, 3, 4, \dots$ **Notation** $a_1, a_2, a_3, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ or $\{ (a_n)_{n \in \mathbb{N}}$

$1, 27, \pi, \sqrt{2}, \dots$

where a_n is the n-th number in the sequence

sometimes (but not always) we can give the sequence by a formula.

Examples $(a_n) = (n)_{n \in \mathbb{N}}$ $1, 2, 3, 4, \dots$

$(a_n) = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ $1, \frac{1}{2}, \frac{1}{3}, \dots$

Example (recursive defn) $a_{n+2} = a_n + a_{n+1}$ $a_1 = 1$ $a_2 = 1$

gives $1, 1, 2, 3, 5, 8, 13, \dots$ (Fibonacci sequence)

Defn A sequence (a_n) converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| < \epsilon$ for all $n \ge N$

Notation $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

Examples $(a_n) = \left(\frac{1}{n}\right)$ $a_n = \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$

check: pick $\epsilon > 0$, then choose $N > \frac{1}{\epsilon}$. if $n > N$ then $\frac{1}{n} < \frac{1}{N} < \epsilon$