§7.6 Strategies for integration

tools: • rewrite expression / simplify. eg: $\dfrac{x^3-1}{x-1} = \dfrac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$.

$$\frac{x-x^3}{\sqrt{x}} = x^{1/2} - x^{5/2}.$$

- substitution
- parts
- trig integrals
- partial fractions.

Examples ① $\int x^3\sqrt{1+x^2}\,dx$ try $u = 1+x^2$, $\dfrac{du}{dx} = 2x$ $\quad \int x^3 . \sqrt{u} . \dfrac{dx}{du}\,du$

$$= \int x^3 \sqrt{u}\,\frac{1}{2x}\,du = \frac{1}{2}\int x^2\sqrt{u}\,du = \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du.$$

② $\int \dfrac{1}{\sqrt{\sqrt{x}+1}}\,dx$ try $u = \sqrt{x}$ ← not simple

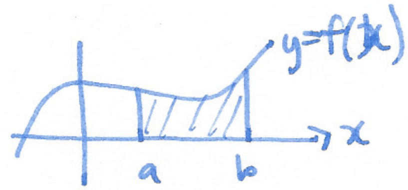
try $u = \sqrt{x}+1$, $\dfrac{du}{dx} = \frac{1}{2}x^{-1/2}$ $\quad \int \dfrac{1}{\sqrt{u}} \cdot \dfrac{1}{\frac{1}{2}\frac{1}{\sqrt{x}}}\,du = 2\int \dfrac{u-1}{\sqrt{u}} \cdot \cancel{\dfrac{1}{u-1}}\,du$

$$= 2\int u^{1/2} - u^{-1/2}\,du.$$

③ $\int \sqrt{x^2+2x+2}\,dx$ complete the square $\int \sqrt{(x+1)^2+1}\,dx$ & trig sub.

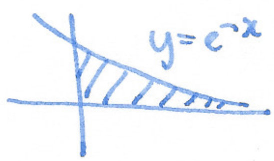
§7.7 Improper integrals

recall $\int_a^b f(x)\,dx =$ area under the curve



Q: what about infinite integrals?

Example

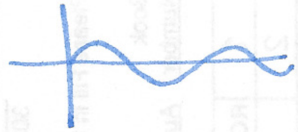


$\int_0^\infty e^{-x}\,dx$ note: $\int_0^R e^{-x}\,dx = \left[-e^{-x}\right]_0^R = -e^{-R} + e^0$

$$= 1 - e^{-R}$$

Defn: $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$, if this limit exists.

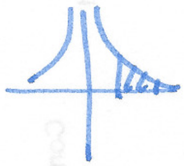
warning: sometimes the limit doesn't exist

Example $\int_0^\infty \sin(x)\,dx$ 

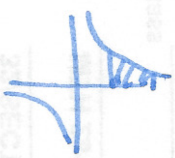
$\int_0^R \sin(x)\,dx = \left[-\cos(x)\right]_0^R = 1-\cos(R)$ $\quad \lim_{R\to\infty} 1-\cos(R)$ DNE!

e.g. $R = 2\pi n$: $1-\cos(R) = 0$

$= 2\pi n + \frac{\pi}{2}$: $1-\cos(R) = 1$

Example ① $\int_1^\infty \frac{1}{x^2}\,dx$  $\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R}+1$

$\lim_{R\to\infty} 1-\frac{1}{R} = 1$

② $\int_1^\infty \frac{1}{x}\,dx$  $= \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \left[\ln|x|\right]_1^R = \ln|R| - \underbrace{\ln|1|}_{0}$

$\to \infty$ as $R \to \infty$.

Doubly infinite integrals $\int_{-\infty}^\infty f(x)\,dx$ (f continuous!) 

Def$^{\text{n}}$ (f cts) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ ← provided each limit exists!

Example $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$

$= \lim_{R\to\infty} \int_{-R}^0 \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_{-R}^0 + \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_0^R$

$= \underbrace{\tan^{-1}(0)}_{0} - \underbrace{\lim_{R\to\infty} \tan^{-1}(-R)}_{-\pi/2} + \underbrace{\lim_{R\to\infty} \tan^{-1}(R)}_{\pi/2} - \underbrace{\tan^{-1}(0)}_{0} = \pi.$

warning $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{R\to\infty} \int_{-R}^R f(x)\,dx$.