so $\int \frac{x^3}{x^2-1}\,dx = \int \frac{x}{x^2-1} + x\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1} + x\,dx$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{2}x^2 + c.$

special case: $\frac{P(x)}{Q(x)}$ $\quad Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i not all distinct, i.e. repeated factors.

Example $\frac{x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ $\quad A, B, C$ numbers.

In general: $\frac{P(x)}{Q(x)}$ if $Q(x) = (x+a_1)^{n_1}(x+a_2)^{n_2}\cdots(x+a_k)^{n_k}$

then $\frac{P(x)}{Q(x)} = \frac{A_{1,1}}{x+a_1} + \frac{A_{1,2}}{(x+a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x+a_1)^{n_1}} + \frac{A_{2,1}}{x+a_2} + \frac{A_{2,2}}{(x+a_2)^2} + \cdots + \frac{A_{k,1}}{(x+a_k)^{n_k}}.$

Example $\frac{x+1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$

equate coefficients: 3 equations in 3 unknowns.

true for all $x \Rightarrow$ true for any particular x:

$x=-2: \quad -1 = C(-3) \Rightarrow C = 1/3$

$x=1: \quad 2 = A(9) \Rightarrow A = 2/9$

$x=0: \quad 1 = 4A-2B-C \Rightarrow 2B = 4A-C-1 = \frac{8}{9} - \frac{1}{3} - 1 = -\frac{4}{9}.$

so $\int \frac{x+1}{(x-1)(x+2)^2}\,dx = \int \frac{2/9}{x-1} + \frac{-2/9}{x+2} + \frac{1/3}{(x+2)^2}\,dx$

$= \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| - \frac{1}{3}\frac{1}{(x+2)} + c.$

special case: quadratic factors:

$$x^2+1 = (x-i)(x+i) \neq (x+a_1)(x+a_2) \quad a_1, a_2 \text{ real.}$$

complex roots!

note: $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

Q: $\int \frac{1}{1+x+x^2}\,dx$? A: complete the square: $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

$$= \int \frac{1}{\frac{3}{4}+(x+1/2)^2}\,dx \quad \text{sub } u = x+\tfrac{1}{2} \quad \frac{du}{dx}=1 \quad \int \frac{1}{3/4+u^2}\,du = \frac{4}{3}\int \frac{1}{1+(\frac{2u}{\sqrt{3}})^2}\,du$$

$$\text{sub } v = \frac{2u}{\sqrt{3}} \quad \frac{dv}{du} = \frac{2}{\sqrt{3}} \quad = \frac{4}{3}\int \frac{1}{1+v^2}\,\frac{\sqrt{3}}{2}\,du = \frac{2}{\sqrt{3}}\tan^{-1}(v)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right)+c.$$

Linear and quadratic factors.

$$\int \frac{1}{x(1+x^2)}\,dx \qquad \frac{1}{x(1+x^2)} = \frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$0x^2+0x+1 = x^2(A+B)+x(C)+A \qquad \left.\begin{array}{l} A+B=0 \\ C=0 \\ A=1 \end{array}\right\} \quad B=-1$$

$$\int \frac{1}{x}+\frac{-x}{x^2+1}\,dx = \ln|x|-\frac{1}{2}\ln|x^2+1|+c.$$

Repeated quadratic factors

$$\frac{1}{(x+a)(x+b)^2(x^2+c)^2} = \frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}+\frac{Dx+E}{x^2+c}+\frac{Fx+G}{(x^2+c)^2}.$$

solve for $A, B, \ldots, G$.

moral: every $P(x)/Q(x)$ can be integrated $\square$.