§7.3 Trig substitutions

aim: deal with $\underset{①}{\sqrt{a^2-x^2}}$, $\underset{②}{\sqrt{a^2+x^2}}$, $\underset{③}{\sqrt{x^2-a^2}}$

use:

$$\left.\begin{array}{l} \cos^2 x+\sin^2 x = 1 \quad \leftrightarrow \quad \sin^2 x = 1-\cos^2 x \quad ① \\ 1+\tan^2 x = \sec^2 x \quad ② \\ \cot^2 x + 1 = \operatorname{cosec}^2 x \quad \leftrightarrow \quad \cot^2 x = \operatorname{cosec}^2 x - 1 \quad ③ \end{array}\right\} \text{convert } \sqrt{} \text{ into } \sqrt{\text{perfect square}}!$$

Example $\int \sqrt{9-x^2}\,dx$ try: $x = 3\sin u$

$$\frac{dx}{du} = 3\cos u$$

$$\int \sqrt{9-(3\sin u)^2}\,\frac{dx}{du}\,du$$

$$\int \sqrt{9-9\sin^2 u}\;3\cos u\,du$$

$$3\int \sqrt{1-\sin^2 u}\cdot 3\cos u\,du$$

$$9\int \sqrt{\cos^2 u}\,\cos u\,du = 9\int \cos^2 u\,du$$

$$\cos 2u = \cos^2 u - \sin^2 u$$
$$= 2\cos^2 u - 1$$
$$\cos^2 u = \tfrac{1}{2}\cos 2u + \tfrac{1}{2}$$

$$= \frac{9}{2}\int \cos 2u + 1\,du = \frac{9}{4}\sin 2u + \frac{9}{2}u + c = \frac{9}{4}2\sin u\cos u + \frac{9}{2}u + c$$

$$= \frac{\cancel{9}}{\cancel{4}}\;\frac{9}{2}\,\frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{9}{2}\sin^{-1}\!\left(\frac{x}{3}\right) + c.$$

Example $\int \sqrt{1+4x^2}\,dx$ try $x = \frac{1}{2}\tan u$

$$\frac{dx}{du} = \frac{1}{2}\sec^2 u$$

$$\int \sqrt{1+4\left(\tfrac{1}{2}\tan u\right)^2}\,\frac{dx}{du}\,du = \int \sqrt{1+\tan^2 u}\ \tfrac{1}{2}\sec^2 u\,du$$

$\int u'v\,dx = uv - \int uv'\,dx$

$$= \int \sqrt{\sec^2 u}\ \tfrac{1}{2}\sec^2 u\,du = \tfrac{1}{2}\int \sec^3 u\,du = \tfrac{1}{2}\int \underbrace{\sec u}_{v}\ \underbrace{\sec^2 u}_{u'}\,du$$

$u' = \sec^2 u \quad u = \tan u$

$v = \sec u \quad v' = \sec u \tan u$

$$= \tfrac{1}{2}\sec u\tan u - \tfrac{1}{2}\int \sec u\tan u\tan u\,du$$

$$\tfrac{1}{2}\int \sec^3 u\,du = \tfrac{1}{2}\sec u\tan u - \tfrac{1}{2}\int \sec u\left(\sec^2 u - 1\right)du.$$

$$\int \sec^3 u\,du = \tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

$\tan u = 2x$

$\sec u = \sqrt{1+4x^2}$

$$= \tfrac{1}{2}\sqrt{1+4x^2}\cdot 2x + \tfrac{1}{2}\ln\left|\sqrt{1+4x^2} + 2x\right| + C$$

Example $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try $x = a\sec u$

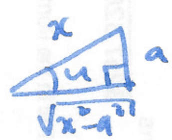
$$\frac{dx}{du} = a\sec u\tan u$$

$$\int \frac{1}{a^2\sec^2 u}\,\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\,\frac{1}{\tan u}\,a\sec u\tan u\,du$$

$$= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + C$$

$x = a\sec u = \frac{a}{\cos u} \qquad \cos u = \frac{a}{x}$

$$= \frac{1}{a^2}\sqrt{1-\left(\tfrac{a}{x}\right)^2} + C.$$



§7.6 Partial fractions

aim: $\int \frac{P(x)}{Q(x)}\,dx$ $\quad P(x), Q(x)$ polynomials

recall: $\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

special case: $\deg(P) < \deg(Q)$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i all distinct real numbers.

then: $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \cdots + \frac{A_n}{x+a_n}$ $\quad A_i$ numbers.

Example $\int \frac{x}{x^2-1}\,dx$ $\qquad x^2-1 = (x-1)(x+1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} = \frac{x(A+B)+(B-A)}{(x+1)(x-1)}$$

so $\left.\begin{array}{l} ① \; A+B=1 \\ ② \; B-A=0 \end{array}\right\}$ $\quad ①+②: \; 2B=1 \quad B=\frac{1}{2}. \quad A=\frac{1}{2}.$

short cut: if $\frac{x}{x^2-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$ works for all x, then it works

for $x=1$: $\; 1 = B(2) \quad B=\frac{1}{2}$

and $x=-1$: $\; -1 = A(-2) \quad A=\frac{1}{2}$.

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$$

special case: as above, but $\deg(P) > \deg(Q)$. long division.

Example $\int \frac{x^3}{x^2-1}\,dx$

$$\begin{array}{r} x \\ x^2-1 \,\big)\, \overline{x^3 } \\ \underline{x^3 - x} \\ x \end{array}$$

↑ remainder.

so $x^3 = (x^2-1)x + x$.