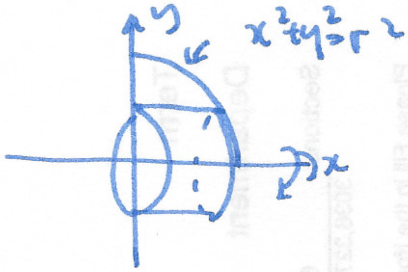
Example



volume of hemisphere: rotate horizontal rectangles around x-axis.

$$V = \int 2\pi y\, f(y)\,dy$$

$$V = \int_0^r 2\pi y\sqrt{r^2-y^2}\,dy = 2\pi\left[\frac{2}{3}\left(r^2-y^2\right)^{3/2}\cdot -\frac{1}{2}\right]_0^r$$

$$= \frac{-2\pi}{3}\left(0-r^3\right) = \frac{2}{3}\pi r^3 \qquad \left(\text{whole sphere } \frac{4}{3}\pi r^3\right)$$

§7.1 Integration by parts

"reverse product rule" product rule: $(uv)' = u'v + uv'$

$$\int (uv)'\,dx = \int u'v\,\cancel{dx} + uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example $\int \underbrace{x}_{u}\underbrace{\sin(x)}_{v'}\,dx$ $\quad u = x$, $u' = 1$ $\quad v' = \sin(x)$, $v = -\cos(x)$

$$\int \underset{u}{x}\,\underset{v'}{\sin(x)}\,dx = \underset{u}{x}\,\underset{v}{(-\cos x)} - \int \underset{u'}{(1)}\,\underset{v}{(-\cos(x))}\,dx$$

$$= -x\cos x + \int \cos(x)\,dx$$

$$= -x\cos x + \sin x + C$$

check!

Examples

1. $$\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx \qquad \left.\begin{matrix} u = x & v' = e^x \\ u' = 1 & v = e^x \end{matrix}\right\}$$ Q: spse we chose them the other way round?

$$= uv - \int u'v\,dx$$

$$= xe^x - \int 1e^x\,dx = xe^x - e^x + C$$ check!

observation: $$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx$$

2. $$\int_1^2 \ln(x)\,dx = \int_1^2 \underbrace{1}_{v'}\cdot\underbrace{\ln(x)}_{u}\,dx \qquad \begin{matrix} u = \ln(x) & v' = 1 \\ u' = \frac{1}{x} & v = x \end{matrix}$$

$$= \left[x\ln(x)\right]_1^2 - \int_1^2 x\cdot\frac{1}{x}\,dx$$

$$= 2\ln(2) - [x]_1^2 = 2\ln(2) - 1$$

3. $$\int \underbrace{x^2}_{u}\underbrace{\cos(x)}_{v'}\,dx = x^2\sin(x) - \int \underbrace{2x}_{u}\,\underbrace{\sin(x)}_{v'}\,dx$$

$$= x^2\sin(x) - 2x(-\cos(x)) + \int 2(-\cos x)\,dx$$

$$= x^2 + 2x\cos(x) - 2\sin x + c$$ (check!).

4. $$\int \underbrace{e^x}_{u}\underbrace{\sin x}_{v'}\,dx = e^x(-\cos x) - \int \underbrace{e^x}_{u}\underbrace{(-\cos x)}_{v'}\,dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) \quad \text{(check!)}$$

§7.2 Trig integrals

$$\int \sin^m x \cos^n x \, dx \; ?$$

techniques:
- $\cos^2 x + \sin^2 x = 1$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- sub $u = \sin x \quad \frac{du}{dx} = \cos x$
- sub $u = \cos x \quad \frac{du}{dx} = -\sin x$
- parts.

Examples

- $\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$ (check!)

- $\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx$ sub $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$= \int (1-u^2) \sin x \cdot \frac{1}{-\sin x} \, du = -\int 1 - u^2 \, du$$

$$= -u + \frac{1}{3}u^3 + c = -\cos x + \frac{1}{3}\cos^3 x + c \quad \text{(check!)}.$$

moral: if there is an odd power, want to do ~~u = sub~~ sub u = other trig function.