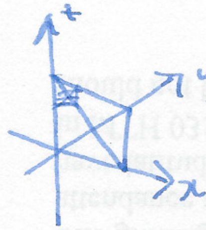
Example: tetrahedron with vertices $(0,0,0)$ $(0,0,1)$ $(0,1,0)$ $(1,0,0)$



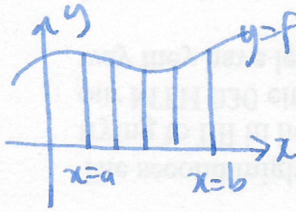
$$V = \int_0^1 A(z)\,dz$$

area of triangle $= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}(1-z)(1-z)$

$$= \int_0^1 \frac{1}{2}(1-z)^2\,dz \cdots = \frac{1}{6}$$

recall: average value of numbers $a_1, \dots, a_n = \frac{a_1 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i$

Q: what about average value of function $f(x)$ on an interval?



average: $\frac{1}{n}\left(f(x_1) + \cdots + f(x_n)\right)$

recall: $\int_a^b f(x)\,dx \approx \left(f(x_1) + \cdots + f(x_n)\right)\Delta x \quad \Delta x = \frac{b-a}{n}$

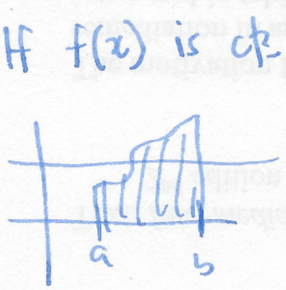
so $\int_a^b f(x)\,dx \approx \text{average} \times (b-a)$ so $\boxed{\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx}$

Example: find average value of $\sin(x)$ on $[0,\pi]$

$[0,\pi]$ $\frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{1}{\pi}\left(-(-1)+1\right) = \frac{2}{\pi}$

useful fact (mean value theorem for integrals)

If $f(x)$ is cts on $[a,b]$ then there is a $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

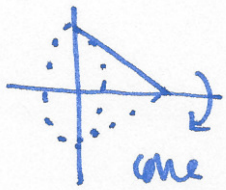


Example find average value of $\frac{\sin(\pi/x)}{x^2}$ on $[1,2]$

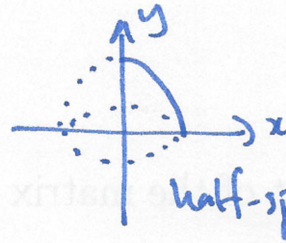
Q: which has bigger average on $[0,\pi]$ $\sin(x)$ or $\sin^2 x$?

§6.3 Volume of revolution

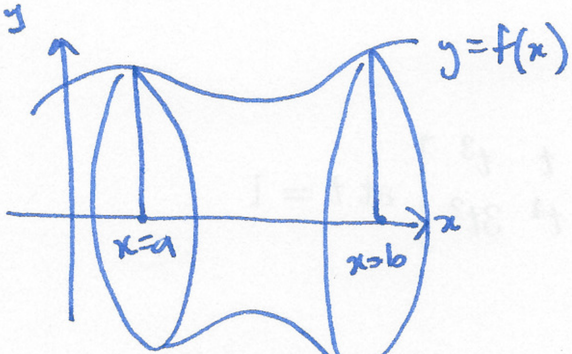
Examples



cone



half-sphere



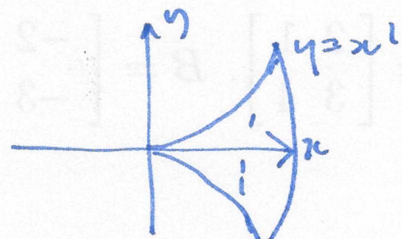
recall: $V = \int_a^b A(x)\,dx$

$A(x)$ = area of vertical cross-section

$= \pi y^2 = \pi (f(x))^2$

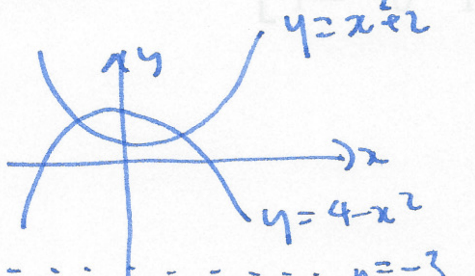
so volume $V = \int_a^b \pi (f(x))^2\,dx$

Example rotate $y = x^2$ about x-axis



$$\pi \int_0^2 (x^2)^2\,dx = \pi \left[\tfrac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$$

Example rotate region between $f(x) = x^2+2$, $g(x) = 4-x^2$ about $y=-3$.



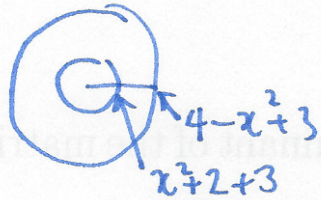
$V = \int_a^b A(x)\,dx$

a, b: find intersections

$x^2+2 = 4-x^2$

$2x^2 = 2 \quad x = \pm 1$

$A(x)$ = area of cross-section:



$$A(x) = \pi\left[(4-x^2+3)^2 - (x^2+2+3)^2\right]$$

$(7-x^2)^2 - (x^2+5)^2$

$49 - 14x^2 + x^4 - x^4 + 10x^2 - 25$

$24 - 4x^2$

$$\pi \int_{-1}^{1} 24 - 4x^2\,dx$$

$$\pi\left[24x - \tfrac{4}{3}x^3\right]_{-1}^{+1} = \left(48 - \tfrac{8}{3}\right)\pi.$$