Math 232 Calculus 2 Fall 17 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes, but no phones or other notes.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Find $\displaystyle\int \frac{1}{x \ln x}\, dx.$

try: $u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$$\int \frac{1}{xu} \frac{dx}{du}\, du = \int \frac{1}{x \cdot u} \cdot x\, du = \int \frac{1}{u}\, du$$

$$= \ln|u| + c = \ln\left|\ln|x|\right| + c$$

(2) Find $\displaystyle\int \frac{x-1}{(x+1)^2}\,dx.$

$$\frac{x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2} \qquad x=-1:\ -2 = B \qquad x=0:\ -1 = A+B \quad A = 1$$

$$\int \frac{1}{x+1} - \frac{2}{(x+1)^2}\,dx = \ln|x+1| + 2(x+1)^{-1} + c$$

$\int uv'\,dx = uv - \int u'v\,dx$

(3) Find $\int_1^{\infty} xe^{-2x}\,dx$.

$\int xe^{-2x}dx = x \cdot -\frac{1}{2}e^{-2x} + \int \frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

$\int_1^{\infty} xe^{-2x}dx = \lim_{R\to\infty} \int_1^R xe^{-2x}dx = \lim_{R\to\infty} \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_1^R$

$= \lim_{R\to\infty} \underbrace{-\frac{1}{2}Re^{-2R} - \frac{1}{4}e^{-2R}}_{\to 0} + \frac{1}{2}e^{-2} + \frac{1}{4}e^{-2} \quad = \frac{3}{4}e^{-2}$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

(4) Find $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$.

try: $x = 3\sin u$

$\frac{dx}{du} = 3\cos u$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{9-9\sin^2 u}} \cdot 3\cos u\,du$$

$$= \int \frac{1}{3\sqrt{\cos^2 u}} 3\cos u\,du = \int 1\,du = u + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{R\to 3} \int_0^R \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{R\to 3} \left[\sin^{-1}\left(\frac{x}{3}\right) \cancel{+C} \right]_0^R$$

$$= \lim_{R\to 3} \sin^{-1}\left(\frac{R}{3}\right) - \sin^{-1}(0) = \frac{\pi}{2}.$$

(5) Find the degree three Taylor polynomial centered at $x = 0$ for the function $f(x) = \sin(e^x)$.

$f(x) = \sin(e^x)$

$f'(x) = \cos(e^x) \cdot e^x$

$f''(x) = \overset{-\sin}{\cancel{\cos}}(e^x) \cdot e^{2x} + \cos(e^x) \cdot e^x$

$f^{(3)}(x) = -\cos(e^x) \cdot e^{3x} + -\sin(e^x) \cdot 2e^{2x} - \sin(e^x) \cdot e^{2x} + \cos(e^x) \cdot e^x$

$f(0) = \sin(1)$

$f'(0) = \cos(1)$

$f''(0) = \cos(1) - \sin(1)$

$f^{(3)}(0) = -3\sin(1)$

$T(x) = \sin(1) + \cos(1)x + \frac{(\cos(1) - \sin(1))x^2}{2!} + (-3\sin(1))\frac{x^3}{3!}$

(6) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$ converge or diverge?

Diverges as $a_n \not\to 0$ $\quad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+3} = \lim_{n\to\infty} \frac{1}{1+3/n} = 1.$

(7) Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Explain your answer.

Diverges: $1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq 1/2} + \cdots$ etc.

(8) (a) Use partial fractions to find an explicit formula for the partial sum

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2 + 3n + 2}.$$

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

$n=-1: \quad 1 = +A$

$n=-2: \quad 1 = -B$

a) $$\frac{1}{n^2+3n+2} = \frac{1}{(n+2)(n+1)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N+1} - \frac{1}{N+2} = \frac{1}{2} - \frac{1}{N+2}$$

b) $$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2} - \frac{1}{N+2} = \frac{1}{2}.$$

(9) (a) Find a formula for the partial sum $s_N = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{3^4} \cdots + \frac{(-1)^N}{3^N}$, for example by comparing s_N with $-\frac{1}{3}s_N$.

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

$$s_N = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \cdots + \frac{(-1)^N}{3^N}$$

$$-\frac{1}{3}s_N = -\frac{1}{9} + \frac{1}{27} - \cdots + \frac{(-1)^N}{3^N} + \frac{(-1)^{N+1}}{3^{N+1}}$$

$$s_N - \left(-\frac{1}{3}s_N\right) = \frac{1}{3} - \frac{(-1)^{N+1}}{3^{N+1}}$$

$$s_N\left(1+\frac{1}{3}\right) = \frac{1}{3} - \frac{(-1)^{N+1}}{3^{N+1}} \qquad s_N = \frac{\frac{1}{3} - \frac{(-1)^{N+1}}{3^{N+1}}}{4/3}.$$

b) $$\lim_{N\to\infty} \frac{1/3 - \frac{(-1)^{N+1}}{3^{N+1}}}{4/3} = \frac{1/3}{4/3} = \frac{1}{4}$$

(10) Show the series $\sum_{n=1}^{\infty} \frac{2^n}{5^n - 2}$ converges, by any method.

limit comparison test: $b_n = \frac{2^n}{5^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{5^n-2} \cdot \frac{5^n}{2^n} = \lim_{n\to\infty} \frac{5^n}{5^n-2} = \lim_{n\to\infty} \frac{1}{1-2/5^n} = 1.$$

$\sum b_n$ converges as geometric series with $|r| < 1$

so $\sum_{n=1}^{\infty} \frac{2^n}{5^n-2}$ converges.