Sample midterm 2 solutions

Q1 $\int \cos^3 x \, dx = \int (1-\sin^2 x)\cos x \, dx$ $\quad u = \sin x \quad \frac{du}{dx} = \cos x$ $\quad \int (1-u^2)\cos x \frac{dx}{du}\, du$

$= \int (1-u^2)\cos x \frac{1}{\cos x}\, du = \int 1 - u^2 \, du = u - \frac{1}{3}u^3 + c = \sin x - \frac{1}{3}\sin^3 x + c$

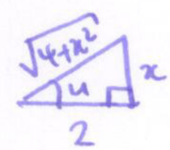
Q2 $\int \cos 4x \sin 3x \, dx$

$$\left.\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}\right\} \begin{aligned} &\sin(A+B)+\sin(A-B) \\ &= 2\sin A\cos B. \end{aligned}$$

$= \frac{1}{2}\int \sin 7x - \sin x \, dx = -\frac{1}{14}\cos 7x + \cos x + c$

$\sin^2 u + \cos^2 u = 1$
$\tan^2 u + 1 = \sec^2 u$

Q3 $\int \frac{x}{\sqrt{x^2+4}}\, dx$ $\quad$ try: $x = 2\tan u \quad \frac{dx}{du} = 2\sec^2 u$ $\quad \int \frac{2\tan u}{\sqrt{4\tan^2 u + 4}} \cdot \frac{dx}{du}\, du$



$\int \frac{2\tan u}{2\sec u} \cdot 2\sec^2 u \, du = 2\int \tan u \sec u \, du = 2\sec u + C = 2 \cdot \frac{\sqrt{4+x^2}}{2} + c$

$= \sqrt{4+x^2} + c.$

Q4 $\frac{3x^2+3x+2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$

$x=-1: \quad 2 = -2C \qquad C = -1$

$x=+1: \quad 8 = A \cdot 4 \qquad A = 2$

$x=0: \quad 2 = A - B - C \qquad B = A - C - 2 = 1$

$\int \frac{2}{x-1} + \frac{1}{x+1} - \frac{1}{(x+1)^2}\, dx = 2\ln|x-1| + \ln|x+1| + \frac{1}{x+1} + c$

Q5 $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx = \underbrace{\frac{1}{3}x^3}_{v}\underbrace{-}_{\ln x} \int \frac{1}{3}x^2 dx = \frac{1}{3}\underbrace{x^3}_{v}\underbrace{-}_{\ln x}\frac{1}{9}x^3 + c$

$\int_0^1 x^2 \ln x \, dx = \lim_{R\to 0}\int_R^1 x^2 \ln x \, dx = \lim_{R\to 0}\left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_R^1$

$= \lim_{R\to 0} -\frac{1}{9} - \frac{1}{3}R^3 \ln R + \frac{1}{9}R^3 \qquad \lim_{R\to 0} R^3 \ln R = \lim_{R\to 0} \frac{\ln R}{R^{-3}} \overset{l'H}{=} \lim_{R\to 0}\frac{1/R}{-3R^{-4}}$

$= \lim_{R\to 0} -\frac{1}{3}R^3 = 0$ so $\lim_{R\to 0} -\frac{1}{9} - \frac{1}{3}R^3 \ln R + \frac{1}{9}R^3 = -\frac{1}{9}$.

Q6 $\int_0^\infty \frac{1}{1+9x^2}\,dx$ $x = \frac{1}{3}\tan u$ $\sin^2 u + \cos^2 u = 1$

$\frac{dx}{du} = \frac{1}{3}\sec^2 u$ $\tan^2 u + 1 = \sec^2 u$.

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+9\left(\frac{1}{3}\tan u\right)^2}\frac{dx}{du}\,du = \int \frac{1}{1+\tan^2 u}\,\frac{1}{3}\sec^2 u \,.\, du$$

$$= \int \frac{1}{3}\,du = \frac{1}{3}u + c = \frac{1}{3}\tan^{-1}(3x) + c$$

$$\int_0^\infty \frac{1}{1+9x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{1+9x^2}\,dx = \lim_{R\to\infty}\left[\frac{1}{3}\tan^{-1}(3x)\right]_0^R$$

$$= \lim_{R\to\infty} \frac{1}{3}\tan^{-1}(3R) - 0 = \frac{\pi}{6}.$$

Q7 $f(x) = (x^2+3)^{1/2}$

$f'(x) = \frac{1}{2}(x^2+3)^{-1/2}.2x$

$f''(x) = -\frac{1}{4}(x^2+3)^{-3/2}.4x^2 + \frac{1}{2}(x^2+3)^{-1/2}.2$

$f^{(3)}(x) = \frac{3}{8}(x^2+3)^{-5/2}.8x^3 - \frac{1}{4}(x^2+3)^{-3/2}.8x - \frac{1}{2}(x^2+3)^{-1/2}.2x$

$f(1) = 2$

$f'(1) = \frac{1}{2}$.

$f''(1) = 1/8 + \frac{1}{2} = \frac{5}{8}$

$f^{(3)}(1) = \frac{3}{32} - \frac{1}{4} - \frac{1}{2}$

$= \frac{-21}{32}$.

$$T_3(\cancel{x-1}) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}}{3!}(x-1)^3$$

$$= 2 + \frac{1}{2}(x-1) + \frac{5}{16}(x-1)^2 - \frac{21}{32\times 6}(x-1)^3.$$

Q8 geometric series: $c + cr + cr^2 + \dots$ $c = \frac{1}{e^2}$ $r = -\frac{1}{e}$.

$$\frac{c}{1-r} = \frac{1/e^2}{1-1/e} = \frac{1}{e^2-e}.$$

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Q4 $\dfrac{2}{n^2+2n} = \dfrac{2}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2} = \dfrac{A(n+2)+Bn}{n(n+2)}$ $n=0:\ 2=2A$, $n=-2:\ 2=-2B$

$= \dfrac{1}{n} - \dfrac{1}{n+2}$ so $\displaystyle\sum_{n=1}^{N} = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots$

$\cdots + \dfrac{1}{N-2} - \dfrac{1}{N} + \dfrac{1}{N-1} - \dfrac{1}{N+1} + \dfrac{1}{N} - \dfrac{1}{N+2}$

$S_N = 1 + \dfrac{1}{2} - \dfrac{1}{N+1} - \dfrac{1}{N+2}$

$\displaystyle\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2} = \frac{3}{2}$. converges to $\frac{3}{2}$.

Q10 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges, integral test with $f(x) = x^2 e^{-\log(2)x}$, or comparison with $\dfrac{1}{(\sqrt{2})^n}$.

integral test: $\displaystyle\int_1^{\infty} x^2 e^{-\ln(2)x}\,dx = x^2 e^{-\ln(2)x} \frac{1}{-\ln(2)} + \int \frac{2x}{\ln(2)} e^{-\ln(2)x}\,dx$

$\displaystyle= \frac{1}{-\ln(2)} x^2 e^{-\ln(2)x} + \frac{2x}{-\ln(2)^2} e^{-\ln(2)x} + \int \frac{2}{\ln(2)^2} e^{-\ln(2)x}\,dx$

$\displaystyle= \frac{x^2 e^{-\ln(2)x}}{-\ln(2)} - \frac{2x e^{-\ln(2)x}}{\ln(2)^2} - \frac{2e^{-\ln(2)x}}{\ln(2)^3}$

$\displaystyle\int_1^{\infty} x^2 e^{-\ln(2)}\,dx = \lim_{R\to\infty} \left[\frac{x^2 e^{-\ln(2)x}}{-\ln(2)} - \frac{2x e^{-\ln(2)x}}{\ln(2)^2} - \frac{2e^{-\ln(2)x}}{\ln(2)^3} \right]_1^R.$

$\displaystyle= \lim_{R\to\infty} -\frac{R^2}{2^R \cdot \ln(2)} - \frac{2R}{2^R \ln(2)^2} - \frac{2}{2^R \ln(2)^3} + \text{const} = \text{const}.$ converges.

comparison: claim: $\exists\, M$ s.t. $n^2 < (\sqrt{2})^n$ for all $n \geq M$.

proof suffices to show $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{2}^x} = 0$ $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^{\log(\sqrt{2})x}} \overset{\text{L'Hô}}{=} \lim_{x\to\infty} \frac{2x}{e^{\log(\sqrt{2})x}\log(\sqrt{2})} = \lim_{x\to\infty} \frac{2}{e^{\log(\sqrt{2})x}\log(\sqrt{2})\log(\sqrt{2})}$

$=0$ $\square$.

Q11 $\displaystyle\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \Rightarrow$ series does not converge.
$(\neq 0)$

Q12 $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^4}$ comparison test: $\frac{\ln(n)^2}{n^2} < 1$ for all $n \geq M$

check: $\lim_{n\to\infty} \frac{\ln(n)^2}{n}$

$= \lim_{n\to\infty} \left(\frac{\ln(n)}{n}\right)^2$

$\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

so $\frac{\ln n}{n^4} < \frac{1}{n^2}$ for all $n \geq M$

$\sum_{n\geq 1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\ln(n)^2}{n^4}$ converges.

Q13: $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$a_n = \overbrace{\underbrace{\frac{2\cdot 2}{1\ 2}}_{4}\ \underbrace{\frac{2\ldots 2}{3\ \ldots}}_{<1}\ \underbrace{\frac{2\ \ 2}{(n-1)(n)}}_{<\frac{4}{(n-1)n}}}^{n} \leq \frac{8}{(n-1)n} \text{ for all } n \geq 3.$$

show $\sum_{n=1} \frac{8}{(n-1)n}$ converges, limit comparison test, with $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{8}{(n-1)n} \cdot n^2 = \lim_{n\to\infty} \frac{8n^2}{n^2-n} = \lim_{n\to\infty} \frac{8}{1-1/n} = 8$, then $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{8}{(n-1)n}$ converges

then comparison test $\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n!}$, converges.

Q14 $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ compare with $b_n = \frac{1}{n^2}$ in limit comparison test

$\lim_{n\to\infty} \frac{x^n}{n^2} \cdot n^2 = \lim_{n\to\infty} x^n = 0$ if $|x| < 1$ so converges as $\sum \frac{1}{n^2}$ converges.

if $|x| > 1$ then $\lim_{n\to\infty} \frac{x^n}{n^2} \neq 0$ so can't converge.

(special cases $x = \pm 1$ converges, $x = +1$ just $\sum_{x=1}^{\infty} \frac{1}{n^2}$)