Math 232 Calculus 2 Fall 17 Final d

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator, and a single US letter page of notes, but no phones or other notes.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) Find $\int \frac{e^x}{\sqrt{e^x+4}}\,dx$.

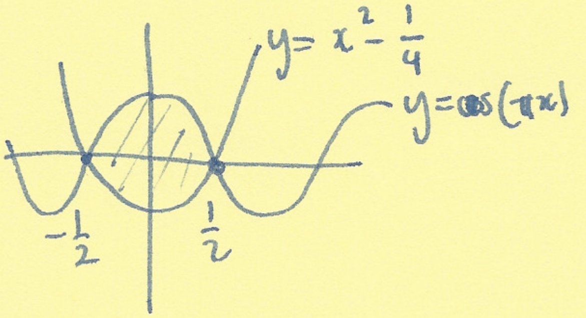
$u = e^x + 4$

$\frac{du}{dx} = e^x$

$$\int \frac{e^x}{\sqrt{u}} \frac{dx}{du}\, du = \int \frac{e^x}{\sqrt{u}} \cdot \frac{1}{e^x}\, du = \int u^{-1/2}\, du = 2u^{1/2} + c$$

$$= 2\sqrt{e^x+4} + c$$

(2) Find the area between the curves $y = \cos(\pi x)$ and $y = (x - 1/2)(x + 1/2)$.



$$\int_{-\frac{1}{2}}^{1/2} \cos(\pi x) - \left(x^2 - \frac{1}{4}\right) dx$$

$$= \left[\frac{1}{\pi}\sin(\pi x) - \frac{1}{3}x^3 + \frac{1}{4}x\right]_{-1/2}^{1/2} = \frac{1}{\pi} - \frac{1}{24} + \frac{1}{8} - \left(-\frac{1}{\pi} + \frac{1}{24} - \frac{1}{8}\right)$$

$$= \frac{2}{\pi} - \frac{1}{12} + \frac{1}{4} = \frac{2}{\pi} + \frac{1}{6} \simeq 0.80$$

(3) Find $\int \sin^3 x \, dx$. $= \int \sin x(1-\cos^2 x)\,dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\int \sin x\,(1-u^2)\frac{dx}{du}\,du = \int \sin x(1-u^2)\frac{1}{-\sin x}\,du = \int -1+u^2\,du = +\frac{1}{3}u^3+c$

$-u$

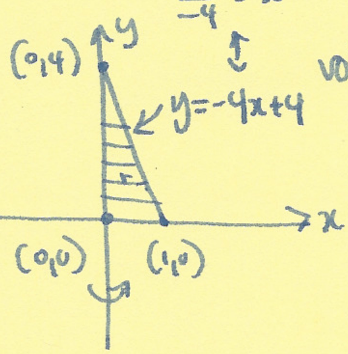
$= +\frac{1}{3}\cos^3 x + C$

$-\cos x$

(4) Find the volume of revolution obtained by rotating the triangle determined by the points $(0,0)$, $(0,4)$ and $(1,0)$ around the y-axis.

$\frac{y-4}{-4} = x$

$\updownarrow$

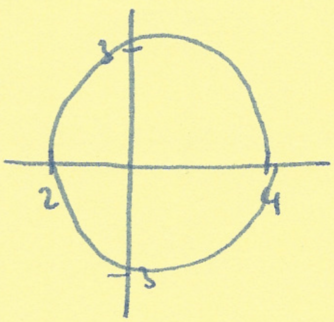


$$\text{vol} = \int \pi r^2 \, dy = \pi \int_0^4 \left(1 - \frac{y}{4}\right)^2 dy$$

$$= \pi \int_0^4 1 - \frac{y}{2} + \frac{y^2}{16} \, dy = \pi \left[y - \frac{y^2}{4} + \frac{y^3}{48} \right]_0^4$$

$$= \pi \left(4 - 4 + \frac{64 \; {\scriptstyle 16\times 4}}{48 \; {\scriptstyle 16\times 3}} \right) = \frac{4\pi}{3}$$

(5) Sketch the polar coordinate graph $r = \cos(\theta) + 3$ and find the area bounded by the curve.



$$\text{area} = \frac{1}{2}\int_0^{2\pi} (\cos\theta + 3)^2 \, d\theta = \frac{1}{2}\int \underbrace{\cos^2\theta}_{\frac{1}{2}\cos 2\theta + \frac{1}{2}} + 6\cos\theta + 9 \, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + 6\sin\theta + 9\theta\right]_0^{2\pi}.$$

$$= \frac{1}{2}\pi + 9\pi = \frac{19}{2}\pi$$

(6) Find the degree three Taylor polynomial for $\sin(e^x)$ centered at $x = 0$.

$$f(x) = \sin(e^x)$$

$$f'(x) = \cos(e^x) . e^x$$

$$f''(x) = -\sin(e^x) . e^{2x} + \cos(e^x) . e^x$$

$$f^{(3)}(x) = -\cos(e^x) . e^{3x} - \sin(e^x) . 2e^{2x} + -\sin(e^x) . e^{2x} + \cos(e^x) . e^x .$$

$$f(0) = \sin(1)$$

$$f'(0) = \cos(1)$$

$$f''(0) = -\sin(1) + \cos(1)$$

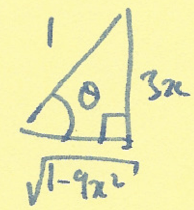
$$f^{(3)}(0) = -3\sin(1)$$

$$T_3(x) = \sin(1) + \cos(1)\, x + \left(\cos(1) - \sin(1)\right) \frac{x^2}{2!} - 3\sin(1) \frac{x^3}{3!}$$

(7) Find $\int \sqrt{1-9x^2}\,dx$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$

$$x = \frac{1}{3}\sin\theta$$
$$\frac{dx}{d\theta} = \frac{1}{3}\cos\theta$$



$$\int \sqrt{1 - 9\frac{1}{9}\sin^2\theta}\,\frac{dx}{d\theta}\,d\theta = \int \sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}\,\frac{1}{3}\cos\theta\,d\theta$$

$$= \int \frac{1}{3}\cos^2\theta\,d\theta = \frac{1}{3}\int \frac{1}{2}\cos 2\theta + \frac{1}{2}\,d\theta = \frac{1}{6}\left[\frac{1}{2}\sin 2\theta + \theta\right]$$

$$= \frac{1}{12}\sin 2\theta + \frac{1}{6}\theta + c = \frac{1}{6}\sin\theta\cos\theta + \frac{1}{6}\theta + c = \frac{1}{6}\cdot x\sqrt{1-9x^2} + \frac{1}{6}\sin^{-1}(3x) + c$$

(8) Find $\int_0^\infty \frac{1}{x^2+8x+17}\,dx$. (Hint: complete the square.)

$(x+4)^2 + 1$

$x^2+8x+16+1$

$\int \frac{1}{(x+4)^2+1}\,dx$ $\quad$ $\cancel{x=u}$ $\quad u = x+4$ $\quad \frac{du}{dx} = 1$

$\int \frac{1}{u^2+1} \frac{dx}{du}\,du = \tan^{-1}(u) + c$

$= \tan^{-1}(x+4) + c$

$\lim_{R\to\infty} \left[\tan^{-1}(x+4) \right]_0^R = \lim_{R\to\infty} \tan^{-1}(R) - \tan^{-1}(4) = \frac{\pi}{2} - \tan^{-1}(4)$

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}.$$

converges, alternating series test

(10) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+4}.$$

diverges, limit comparison test w/ $b_n = \frac{1}{\sqrt{n}}$

(11) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{n}{8^n}.$$

converges, comparison test w $\frac{n}{8^n} < \frac{1}{4^n}$ for $n >> 1$.

(12) Find the power series for $\dfrac{e^x - 1}{x}$, centered at $x = 0$. You may use the power series for e^x without justification. Use this to find a power series for $\displaystyle\int \frac{e^x - 1}{x}\,dx$. What is the radius of convergence for this power series?

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad a_n = \frac{x^n}{n!}$

$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \qquad a_n = \frac{x^{n}}{(n+1)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$

$\Rightarrow R = \infty.$