Math 232 Calculus 2 Fall 17 Final a

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator, and a single US letter page of notes, but no phones or other notes.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) Find $\int \frac{e^x}{\sqrt{e^x+3}}\,dx.$

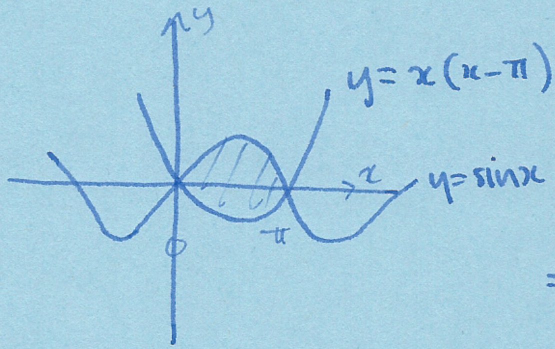
$u = e^x + 3$

$\frac{du}{dx} = e^x$

$$\int \frac{e^x}{\sqrt{u}} \cdot \frac{dx}{du}\,du = \int \frac{e^x}{\sqrt{u}} \cdot \frac{1}{e^x}\,du = \int u^{-1/2}\,du$$

$$= 2u^{1/2} + c = 2\sqrt{e^x+3} + c$$

(2) Find the area between the curves $y = \sin x$ and $y = x(x - \pi)$.



$$-\int_0^{\pi} x^2 - \pi x - \sin x \, dx$$

$$= -\left[\frac{1}{3}x^3 - \frac{\pi}{2}x^2 + \cos x \right]_0^{\pi}$$

$$= -\frac{1}{3}\pi^3 + \frac{1}{2}\pi^3 + 1 + (1) = \frac{\pi^3}{6} + 2 \approx 7.17$$

(3) Find $\int \cos^3 x \, dx$. $= \int \cos x (1-\sin^2 x) \, dx$

$u = \sin x$

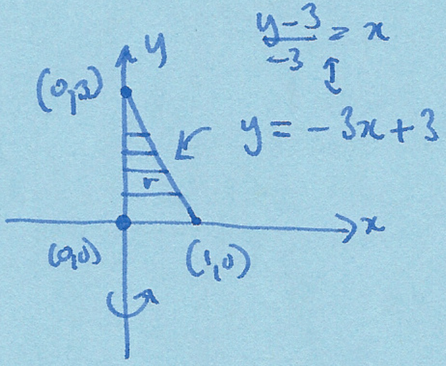
$\frac{du}{dx} = \cos x$

$$\int \cos x (1-u^2) \frac{dx}{du} \, du = \int \cos x (1-u^2) \frac{1}{\cos x} \, du = \int 1-u^2 \, du = u - \frac{1}{3}u^3 + c$$

$\cancel{= \frac{1}{3}\sin^3 x + c}$

$\sin x - \frac{1}{3}\sin^3 x + c$

(4) Find the volume of revolution obtained by rotating the triangle determined by the points $(0,0)$, $(0,3)$ and $(1,0)$ around the y-axis.

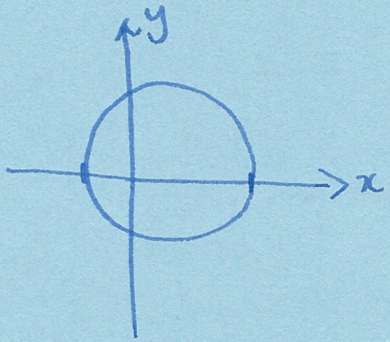


$$\text{vol} = \int \pi r^2 \, dy = \pi \int_0^3 \left(1 - \frac{y}{3}\right)^2 dy$$

$$= \pi \int_0^3 1 - \frac{2y}{3} + \frac{y^2}{9} \, dy$$

$$= \pi \left[y - \frac{y^2}{3} + \frac{y^3}{27} \right]_0^3 = \pi \left(3 - 3 + 1 \right) = \pi$$

(5) Sketch the polar coordinate graph $r = \cos(\theta) + 2$ and find the area bounded by the curve.



$$\int_0^{2\pi} \frac{1}{2}(\cos\theta+2)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \cos^2\theta + 4\cos\theta + 4\, d\theta$$

$$\cos 2\theta = \cos^2\theta - \underbrace{\sin^2\theta}_{\parallel \atop 1-\cos^2\theta}$$

$$= 2\cos^2\theta - 1$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2}\cos 2\theta + \frac{1}{2} + 4\cos\theta + 4\, d\theta = \frac{1}{2}\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta - 4\sin\theta + 4\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(0 + \pi - 0 + 8\pi\right) = \frac{9}{2}\pi$$

(6) Find the degree three Taylor polynomial for $\sin(e^x)$ centered at $x = 0$.

$f(x) = \sin(e^x)$

$f'(x) = \cos(e^x) \cdot e^x$

$f''(x) = -\sin(e^x) \cdot e^{2x} + \cos(e^x) \cdot e^x$

$f^{(3)}(x) = -\cos(e^x) \cdot e^{2x} - \sin(e^x) \cdot 2e^{2x} - \sin(e^x) \cdot e^{2x} + \cos(e^x) e^x$

$f(0) = \sin(1)$

$f'(0) = \cos(1)$

$f''(0) = -\sin(1) + \cos(1)$

$f^{(3)}(0) = -\cos(1) - \sin(1) - \sin(1) + \cos(1) = -2\sin(1)$

$T_3(x) = \sin(1) + \cos(1)x + (\cos(1) - \sin(1))\frac{x^2}{2!} - 2\sin(1)\frac{x^3}{3!}$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$

(7) Find $\int \sqrt{9-x^2}\,dx$.

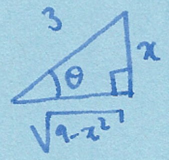
$$x = 3\sin\theta$$
$$\frac{dx}{d\theta} = 3\cos\theta$$

$$\int \sqrt{9 - 9\sin^2\theta}\,\frac{dx}{d\theta}\,d\theta = \int 3\sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}\;3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta$$

$$\cos 2\theta = \cos^2\theta - \underbrace{\sin^2\theta}_{1-\cos^2\theta}$$
$$= 2\cos^2\theta - 1$$

$$= 9\int \frac{1}{2}\cos 2\theta + \frac{1}{2}\,d\theta = \frac{9}{2}\left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right] = \frac{9}{4}\sin 2\theta + \frac{9}{4}\theta + c$$

$$= \frac{9}{4}2\sin\theta\cos\theta + \frac{9}{4}\theta + c = \frac{9}{2}x\,\frac{\sqrt{9-x^2}}{3} + \frac{9}{4}\sin^{-1}\left(\frac{x}{3}\right) + c$$



$$= \frac{3}{2}x\sqrt{9-x^2} + \frac{9}{4}\sin^{-1}\left(\frac{x}{3}\right) + c$$

(8) Find $\int_0^\infty \frac{1}{x^2+4x+5}\,dx$. (Hint: complete the square.)

$(x+2)^2+1$

$(x^2+4x+4)+1$

$\int \frac{1}{(x+2)^2+1}\,dx$ $\quad u=x+2$, $\frac{du}{dx}=1$ $\quad \int \frac{1}{u^2+1}\,du = \tan^{-1}(u)+c$

$\lim_{R\to\infty} \left[\tan^{-1}(x+2)\right]_0^R = \lim_{R\to\infty} \tan^{-1}(R+2)-\tan^{-1}(2) = \frac{\pi}{2}-\tan^{-1}(2)$

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}.$$

$(-1)^n a_n \qquad a_n = \frac{1}{\ln(n)}$

converges by alternating series test

check : $a_n = \frac{1}{\ln(n)}$ positive ✓

decreasing: $\ln(x)$ monotonic so

$n < n+1$

$\ln(n) < \ln(n+1)$

$\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$

$a_n > a_{n+1}$ as required.

$\lim_{n \to \infty} \frac{1}{\ln(n)} = 0.$

(10) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+3}.$$

diverges: limit ~~ratio~~ comparison test with $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{1}{\sqrt{n}+3} \cdot \sqrt{n}\right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+3}$$

$$= \lim_{n\to\infty} \frac{1}{1+3/\sqrt{n}} = 1 \quad (\text{not } 0, \infty) \Rightarrow \sum a_n \text{ converges iff } \sum b_n \text{ converges}$$

$\sum \frac{1}{\sqrt{n}}$ diverges (p-series $p \leq 1$) $\Rightarrow$ $\sum \frac{1}{\sqrt{n}+3}$ diverges.

(11) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{n}{4^n}.$$

$n < 2^n$ for all $n \geq 2$

so $\frac{n}{4^n} < \frac{1}{2^n}$

comparison test $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, geometric series w/ $r < 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{4^n}$ converges.

N Limit ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n} = \lim_{n\to\infty} \frac{1}{4} \frac{n+1}{n}$

$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right) = \frac{1}{4} < 1 \Rightarrow$ converges.

(12) Find the power series for $\dfrac{1-e^x}{x}$, centered at $x = 0$. You may use the power series for e^x without justification. Use this to find a power series for $\displaystyle\int \frac{1-e^x}{x}\,dx$. What is the radius of convergence for this power series?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad a_n = \frac{x^n}{n!}$$

$$1 - e^x = \quad -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots$$

$$\frac{1-e^x}{x} = \quad -1 - \frac{x}{2!} - \frac{x^2}{3!} - \cdots \qquad a_n = -\frac{x^n}{(n+1)!}$$

ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+2} = 0$

converges for all $x \Rightarrow$ radius of convergence $R = \infty$.